Exercise 5

(a) Find the steady-state temperature distribution inside an annular plate $\{1 < r < 2\}$, whose outer edge ($r = 2$) is insulated, and on whose inner edge ($r = 1$) the temperature is maintained as $\sin^2\theta$. (Find explicitly all the coefficients, etc.)

(b) Same, except $u = 0$ on the outer edge.

Solution

The governing equation for the steady-state temperature u in a domain without heat sources is the Laplace equation.

$$\nabla^2 u = 0$$

Since the domain we want to solve it in is an annulus ($1 < r < 2$), we choose to write the Laplacian operator in polar coordinates.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 1 < r < 2,\ 0 < \theta < 2\pi \tag{1}$$

Part (a)

The insulation at $r = 2$ means that the derivative in the r-direction (normal to the circular boundary) is zero. The temperature at $r = 1$ is given to be $\sin^2\theta$, which can be written as $\frac{1}{2}(1 - \cos 2\theta)$.

$$u(1,\theta) = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$
$$u_r(2,\theta) = 0$$

From the form of the inhomogeneous boundary condition we hypothesize that the solution has the form

$$u(r,\theta) = \frac{1}{2} + g(r)\cos 2\theta.$$

Apply the boundary conditions for u to determine the boundary conditions for g.

$$\begin{aligned} u(1,\theta) &= \frac{1}{2} + g(1)\cos 2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta \quad &\rightarrow \quad g(1) = -\frac{1}{2} \\ u_r(2,\theta) &= g'(2)\cos 2\theta = 0 \quad &\rightarrow \quad g'(2) = 0 \end{aligned} \tag{2}$$

In order to determine $g(r)$, substitute the expression for $u(r,\theta)$ into equation (1).

$$\frac{\partial^2}{\partial r^2}\left[\frac{1}{2} + g(r)\cos 2\theta\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{2} + g(r)\cos 2\theta\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left[\frac{1}{2} + g(r)\cos 2\theta\right] = 0$$

Evaluate the derivatives.

$$g''(r)\cos 2\theta + \frac{1}{r}g'(r)\cos 2\theta + \frac{1}{r^2}g(r)(-4\cos 2\theta) = 0$$

Multiply both sides by $r^2/\cos 2\theta$.

$$r^2 g'' + r g' - 4g = 0$$

Since θ is not present in this equation, the hypothesis for $u(r, \theta)$ is legitimate. This is an equidimensional ODE for g, so it has solutions of the form

$$g(r) = r^m \quad \rightarrow \quad g'(r) = mr^{m-1} \quad \rightarrow \quad g''(r) = m(m-1)r^{m-2}.$$

Substitute these expressions into the ODE to determine the constants m.

$$m(m-1)r^m + mr^m - 4r^m = 0$$

Divide both sides by r^m.

$$m(m-1) + m - 4 = 0$$

Solve for m.

$$m^2 - 4 = 0 \quad \rightarrow \quad m = \{\pm 2\}$$

Consequently,

$$g(r) = C_1 r^2 + C_2 r^{-2}.$$

Now apply the boundary conditions for g to determine C_1 and C_2.

$$g(1) = C_1 + C_2 = -\frac{1}{2}$$
$$g'(2) = 4C_1 - \frac{C_2}{4} = 0$$

Solving the system of equations yields $C_1 = -1/34$ and $C_2 = -8/17$.

$$\begin{aligned} g(r) &= -\frac{1}{34} r^2 - \frac{8}{17} r^{-2} \\ &= -\frac{r^4 + 16}{34r^2} \end{aligned}$$

Therefore,

$$u(r, \theta) = \frac{1}{2} - \frac{r^4 + 16}{34r^2} \cos 2\theta.$$

This solution can be written in Cartesian coordinates by writing $\cos 2\theta = 2\cos^2\theta - 1$ and then using $r^2 = x^2 + y^2$ and $\cos\theta = x/\sqrt{x^2 + y^2}$.

$$\begin{aligned} u(x, y) &= \frac{1}{2} - \frac{(x^2 + y^2)^2 + 16}{34(x^2 + y^2)} \left(\frac{2x^2}{x^2 + y^2} - 1 \right) \\ &= \frac{1}{2} - \frac{(x^2 + y^2)^2 + 16}{34(x^2 + y^2)^2} (x^2 - y^2) \end{aligned}$$

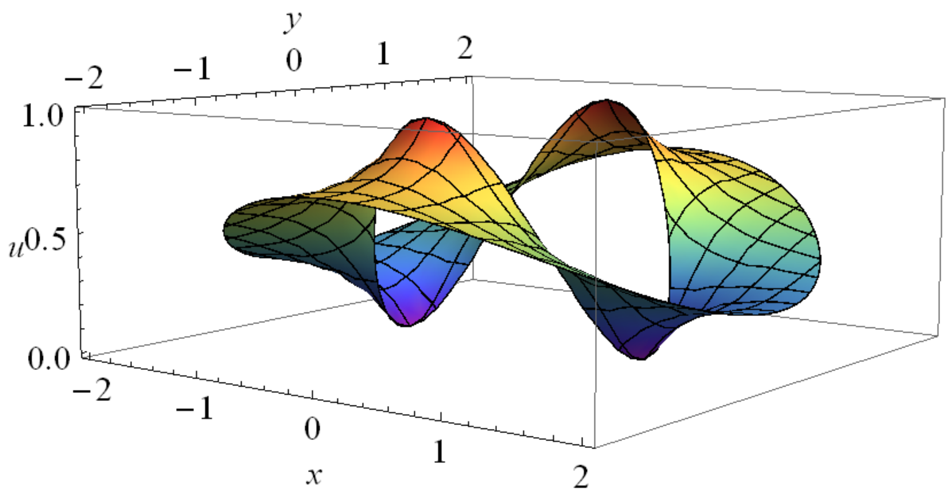

Figure 1: This is a plot of the two-dimensional solution surface $u(x, y)$ in three-dimensional xyu-space. Notice that the maximum and minimum values of u lie on the boundary (maximum principle).

Part (b)

Here the boundary conditions are

$$u(1,\theta) = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$
$$u(2,\theta) = 0.$$

From the form of the inhomogeneous boundary condition we hypothesize that the solution has the form

$$u(r,\theta) = f(r) + h(r)\cos 2\theta.$$

Apply the boundary conditions for u to determine the boundary conditions for f and h.

$$u(1,\theta) = f(1) + h(1)\cos 2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta \quad \rightarrow \quad f(1) = \frac{1}{2} \quad \text{and} \quad h(1) = -\frac{1}{2}$$
$$u(2,\theta) = f(2) + h(2)\cos 2\theta = 0 \quad \rightarrow \quad f(2) = 0 \quad \text{and} \quad h(2) = 0$$

In order to determine $f(r)$ and $h(r)$, substitute the expression for $u(r,\theta)$ into equation (1).

$$\frac{\partial^2}{\partial r^2}[f(r) + h(r)\cos 2\theta] + \frac{1}{r}\frac{\partial}{\partial r}[f(r) + h(r)\cos 2\theta] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}[f(r) + h(r)\cos 2\theta] = 0$$

Evaluate the derivatives.

$$f''(r) + h''(r)\cos 2\theta + \frac{1}{r}[f'(r) + h'(r)\cos 2\theta] + \frac{1}{r^2}[-4h(r)\cos 2\theta] = 0$$

Expand the left side.

$$f''(r) + \frac{1}{r}f'(r) + h''(r)\cos 2\theta + \frac{1}{r}h'(r)\cos 2\theta - \frac{4}{r^2}h(r)\cos 2\theta = 0$$

If we set

$$f''(r) + \frac{1}{r}f'(r) = 0, \tag{3}$$

then the previous equation reduces to

$$h''(r)\cos 2\theta + \frac{1}{r}h'(r)\cos 2\theta - \frac{4}{r^2}h(r)\cos 2\theta = 0.$$

Dividing both sides by $\cos 2\theta$,

$$h''(r) + \frac{1}{r}h'(r) - \frac{4}{r^2}h(r) = 0, \tag{4}$$

we obtain an equation that is independent of θ, proving the legitimacy of the hypothesis. Equation (3) is first-order in f', so we multiply both sides by the integrating factor I.

$$I = \exp\left(\int^r \frac{1}{s}\,ds\right) = \exp(\ln r) = r$$

Doing so gives us

$$rf'' + f' = 0.$$

The left side can be written as $d/dr(If')$ as a result of the product rule.

$$\frac{d}{dr}(rf') = 0$$

Integrate both sides with respect to r.

$$rf' = C_3$$

Divide both sides by r.

$$f' = \frac{C_3}{r}$$

Integrate both sides with respect to r once more.

$$f(r) = C_3 \ln r + C_4$$

Apply the two boundary conditions for f to determine C_3 and C_4.

$$f(1) = C_4 = \frac{1}{2}$$

$$f(2) = C_3 \ln 2 + C_4 = 0 \quad \rightarrow \quad C_3 = -\frac{1}{2\ln 2}$$

Consequently,

$$\begin{aligned} f(r) &= -\frac{1}{2\ln 2}\ln r + \frac{1}{2} \\ &= \frac{1}{2}\left(1 - \frac{\ln r}{\ln 2}\right). \end{aligned}$$

Now equation (4) will be solved. Multiply both sides of it by r^2.

$$r^2 h'' + rh' - 4h = 0$$

This is identical to the ODE solved earlier for g, so it has the same general solution.

$$h(r) = C_5 r^2 + C_6 r^{-2}$$

Apply the two boundary conditions for h to determine C_5 and C_6.

$$h(1) = C_5 + C_6 = -\frac{1}{2}$$
$$h(2) = 4C_5 + \frac{C_6}{4} = 0$$

Solving this system of equations yields $C_5 = 1/30$ and $C_6 = -8/15$. Consequently,

$$\begin{aligned} h(r) &= \frac{1}{30}r^2 - \frac{8}{15r^2} \\ &= \frac{r^4 - 16}{30r^2}. \end{aligned}$$

Therefore,

$$u(r,\theta) = \frac{1}{2}\left(1 - \frac{\ln r}{\ln 2}\right) + \frac{r^4 - 16}{30r^2}\cos 2\theta.$$

This solution can be written in Cartesian coordinates by writing $\cos 2\theta = 2\cos^2\theta - 1$ and then using $r^2 = x^2 + y^2$ and $\cos\theta = x/\sqrt{x^2+y^2}$.

$$\begin{aligned} u(x,y) &= \frac{1}{2}\left(1 - \frac{\ln\sqrt{x^2+y^2}}{\ln 2}\right) + \frac{(x^2+y^2)^2 - 16}{30(x^2+y^2)}\left(\frac{2x^2}{x^2+y^2} - 1\right) \\ &= \frac{1}{2}\left(1 - \frac{\ln\sqrt{x^2+y^2}}{\ln 2}\right) + \frac{(x^2+y^2)^2 - 16}{30(x^2+y^2)^2}(x^2 - y^2) \end{aligned}$$

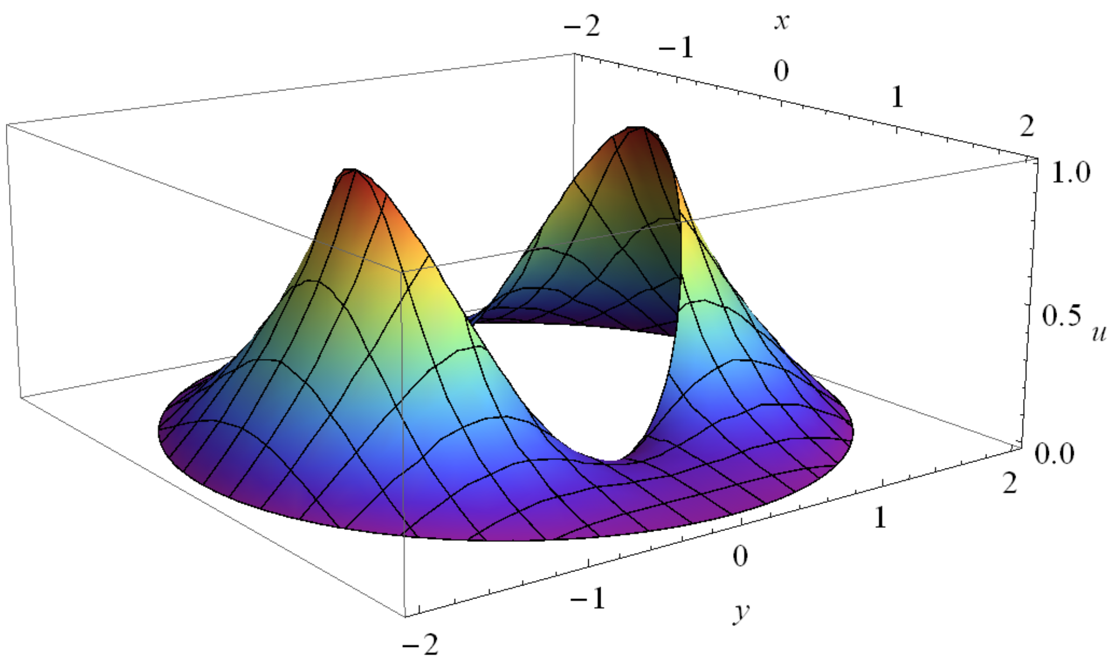

Figure 2: This is a plot of the two-dimensional solution surface $u(x,y)$ in three-dimensional xyu-space. Notice that the maximum and minimum values of u lie on the boundary (maximum principle).