Exercise 8

An annular plate with inner radius a and outer radius b is held at temperature B at its outer boundary and satisfies the boundary condition $\partial u/\partial r = A$ at its inner boundary, where A and B are constants. Find the temperature if it is at a steady state. (*Hint:* It satisfies the two-dimensional Laplace equation and depends only on r.)

Solution

The governing equation for the steady-state temperature u in a domain without heat sources is the Laplace equation.

$$\nabla^2 u = 0$$

Since the domain we want to solve it in is an annulus, we choose to write the Laplacian operator in polar coordinates. We also expect the solution to be the same in value and slope (in the θ-direction) at $\theta = 0$ as it is at $\theta = 2\pi$.

$$\begin{gathered}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad a < r < b,\ 0 < \theta < 2\pi \\ u_r(a,\theta) = A, \quad u(r,0) = u(r,2\pi) \\ u(b,\theta) = B, \quad u_\theta(r,0) = u_\theta(r,2\pi)\end{gathered}$$

Because the boundary conditions at $r = a$ and $r = b$ are independent of θ, we assume that u is only a function of r, $u = u(r)$. As a result, the Laplace equation simplifies to an ODE,

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = 0,$$

with boundary conditions,

$$\begin{aligned}\frac{du}{dr}(a) &= A \\ u(b) &= B.\end{aligned}$$

The ODE is first-order in du/dr, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^r \frac{1}{s}\,ds\right) = \exp(\ln r) = r$$

Doing so gives us

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} = 0.$$

The left side can be written as $d/dr(I du/dr)$ by the product rule.

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{du}{dr} = C_1$$

Divide both sides by r.

$$\frac{du}{dr} = \frac{C_1}{r}$$

Apply the boundary condition at $r = a$ to determine C_1.

$$\frac{du}{dr}(a) = \frac{C_1}{a} = A \quad \rightarrow \quad C_1 = aA$$

So then

$$\frac{du}{dr} = \frac{aA}{r}.$$

Integrate both sides with respect to r once more.

$$u(r) = aA \ln r + C_2$$

Apply the boundary condition at $r = b$ to determine C_2.

$$u(b) = aA \ln b + C_2 = B \quad \rightarrow \quad C_2 = B - aA \ln b$$

We then have

$$\begin{aligned} u(r) &= aA \ln r + B - aA \ln b \\ &= B + aA(\ln r - \ln b). \end{aligned}$$

Therefore,

$$u(r) = B + aA \ln \frac{r}{b}.$$