Exercise 9

Solve $u_{xx} + u_{yy} = 0$ in the wedge $r < a,\ 0 < \theta < \beta$ with the BCs

$$u = \theta \quad \text{on } r = a, \quad u = 0 \quad \text{on } \theta = 0, \quad \text{and} \quad u = \beta \quad \text{on } \theta = \beta.$$

(*Hint:* Look for a function independent of r.)

Solution

The domain is a wedge, so we choose to expand the Laplacian operator in polar coordinates. The boundary value problem to solve then is the following.

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, &\quad r < a, 0 < \theta < \beta \\ u(0,\theta) = \text{bounded}, &\quad u(r,0) = 0 \\ u(a,\theta) = \theta, &\quad u(r,\beta) = \beta \end{aligned}$$

From the boundary conditions we assume that the solution is only a function of θ, $u = u(\theta)$. As a result, the Laplace equation simplifies to an ODE,

$$\frac{1}{r^2}\frac{d^2u}{d\theta^2} = 0 \quad \rightarrow \quad \frac{d^2u}{d\theta^2} = 0,$$

with boundary conditions,

$$\begin{aligned} u(0) &= 0 \\ u(\beta) &= \beta. \end{aligned}$$

Integrate both sides of the ODE with respect to θ.

$$\frac{du}{d\theta} = C_1$$

Integrate both sides of the ODE with respect to θ once more.

$$u(\theta) = C_1\theta + C_2$$

Apply the boundary conditions here to determine C_1 and C_2.

$$\begin{aligned} u(0) &= C_2 = 0 \\ u(\beta) &= C_1\beta + C_2 = \beta \end{aligned}$$

Solving the second equation for C_1 gives $C_1 = 1$. Therefore,

$$u(\theta) = \theta.$$

In Cartesian coordinates this is

$$u(x,y) = \tan^{-1}\frac{y}{x}.$$

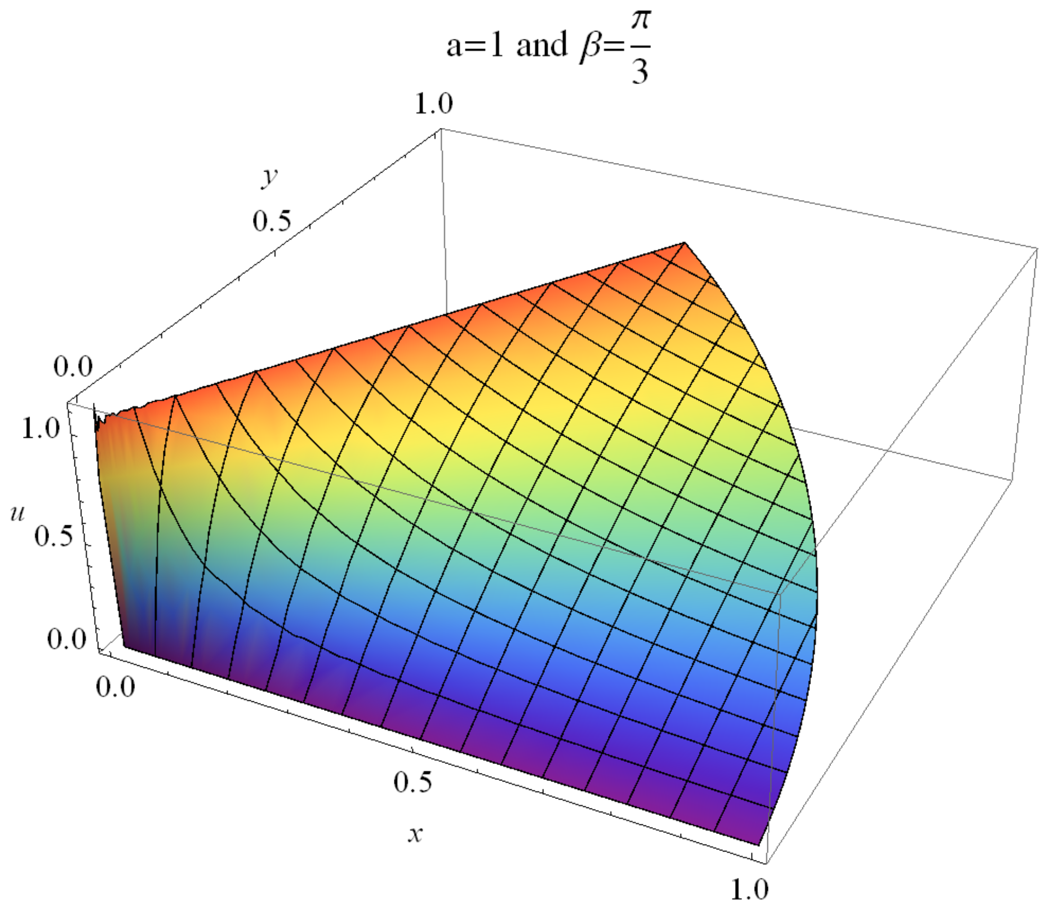

Figure 1: This is a plot of the two-dimensional solution surface $u(x, y)$ in three-dimensional xyu-space for $a = 1$ and $\beta = \pi/3$. Notice that the maximum and minimum values of u lie on the boundary (maximum principle).