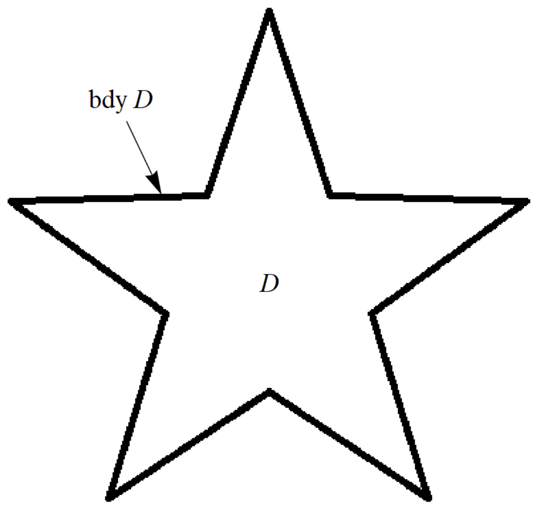
Exercise 1

Derive the three-dimensional maximum principle from the mean value property.

Solution

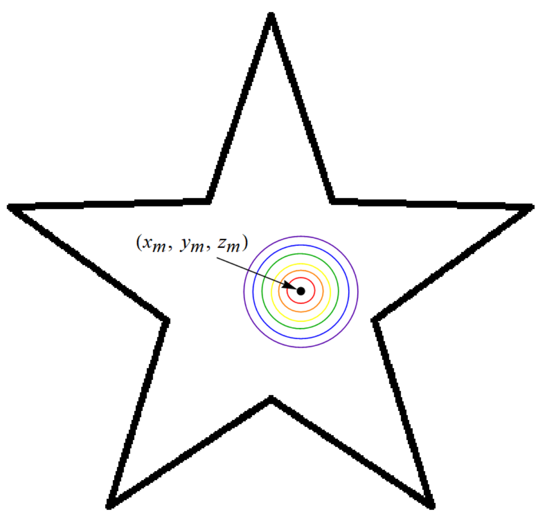
We want to prove the maximum principle, which reads as follows: If D is any solid (connected) region, a nonconstant harmonic function in D cannot take its maximum value inside D.



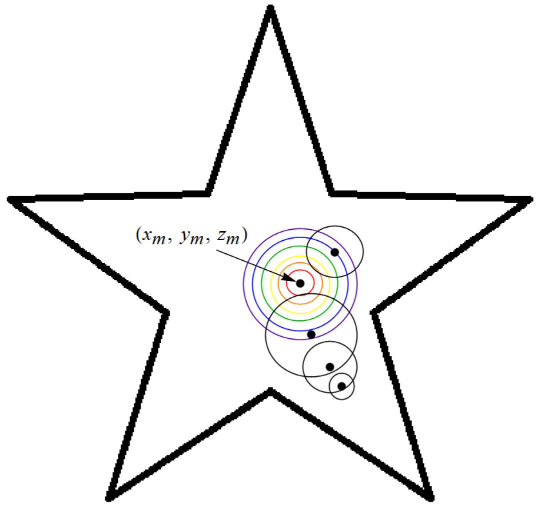
Let u be a nonconstant harmonic function of x, y, and z in D. Then u satisfies the Laplace equation in D.

$$\nabla^2 u = 0 \quad \text{in } D$$

Because u is continuous, its maximum value must occur either within D or on the boundary of D. Suppose that the maximum is located at $(x_m, y_m, z_m) \in D$, and let the value of u there be $u(x_m, y_m, z_m) = M$. The aim in this exercise is to show that a contradiction results and that $(x_m, y_m, z_m) \notin D$. Consider a series of spheres centered at the maximum as illustrated below.



According to the mean value property, the average value of u over each of these spheres is equal to M, its value at the center. But since M is the maximum in D, no higher values can occur on any particular sphere. No lower values than M can occur either because otherwise the average would be less than M. Thus, $u = M$ on each sphere lying in D centered at (x_m, y_m, z_m). By considering spheres centered at points on the edge of where $u = M$, it follows that $u = M$ in all of D.



However, this contradicts the assumption that u is nonconstant. Therefore, the maximum value of a nonconstant harmonic function cannot occur inside D.