Exercise 2

Prove the uniqueness up to constants of the Neumann problem using the energy method.

Solution

The Neumann problem is

$$\begin{aligned} \Delta u &= f \quad \text{in } D \\ \frac{\partial u}{\partial n} &= h \quad \text{on bdy } D. \end{aligned}$$

Suppose that in addition to u there is a second solution v to this problem.

$$\begin{aligned} \Delta v &= f \quad \text{in } D \\ \frac{\partial v}{\partial n} &= h \quad \text{on bdy } D \end{aligned}$$

Subtract the respective sides of the equations valid in D as well as the respective sides of the equations valid on bdy D.

$$\begin{aligned} \Delta u - \Delta v &= f - f \quad \text{in } D \\ \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} &= h - h \quad \text{on bdy } D \end{aligned}$$

Factor the operator in the first equation.

$$\begin{aligned} \Delta(u - v) &= 0 \quad \text{in } D \\ \frac{\partial}{\partial n}(u - v) &= 0 \quad \text{on bdy } D \end{aligned}$$

Let $w = u - v$.

$$\begin{aligned} \Delta w &= 0 \quad \text{in } D \\ \frac{\partial w}{\partial n} &= 0 \quad \text{on bdy } D \end{aligned}$$

Taking the two arbitrary functions to be w, Green's first identity says that

$$\begin{aligned} \iint\limits_{\text{bdy } D} w \frac{\partial w}{\partial n}\, dS &= \iiint\limits_{D} |\nabla w|^2\, dV + \iiint\limits_{D} w \Delta w\, dV \\ 0 &= \iiint\limits_{D} |\nabla w|^2\, dV. \end{aligned}$$

By the vanishing theorem, the integrand is zero.

$$|\nabla w|^2 = 0$$

$$\nabla w = \mathbf{0}$$

$$w = \text{constant}$$

Therefore, $u = v + \text{constant}$, which means the solution to the Neumann problem is unique up to a constant.