Exercise 4

Generalize the energy method to prove uniqueness for the diffusion equation with Dirichlet boundary conditions in three dimensions.

Solution

The diffusion equation subject to a Dirichlet boundary condition and an initial condition is

$$\begin{aligned} u_t &= k\Delta u + f && \text{in } D \\ u &= g && \text{in } D \text{ at } t = 0 \\ u &= h && \text{on bdy } D. \end{aligned}$$

Suppose that in addition to u there is a second solution v to this problem.

$$\begin{aligned} v_t &= k\Delta v + f && \text{in } D \\ v &= g && \text{in } D \text{ at } t = 0 \\ v &= h && \text{on bdy } D \end{aligned}$$

Subtract the respective sides of the equations valid in D as well as the respective sides of the equations valid on bdy D.

$$\begin{aligned} u_t - v_t &= k\Delta u + f - k\Delta v - f && \text{in } D \\ u - v &= g - g && \text{in } D \text{ at } t = 0 \\ u - v &= h - h && \text{on bdy } D \end{aligned}$$

Factor the operators in the first equation.

$$\begin{aligned} (u - v)_t &= k\Delta(u - v) && \text{in } D \\ u - v &= 0 && \text{in } D \text{ at } t = 0 \\ u - v &= 0 && \text{on bdy } D \end{aligned}$$

Let $w = u - v$.

$$\begin{aligned} w_t &= k\Delta w && \text{in } D && (1) \\ w &= 0 && \text{in } D \text{ at } t = 0 \\ w &= 0 && \text{on bdy } D \end{aligned}$$

Multiply both sides of equation (1) by w.

$$ww_t = kw\Delta w$$

Rewrite the left side as a derivative.

$$\frac{1}{2}\frac{\partial}{\partial t}(w^2) = kw\Delta w$$

Integrate both sides over the volume of D.

$$\iiint_D \frac{1}{2}\frac{\partial}{\partial t}(w^2)\, dV = \iiint_D kw\Delta w\, dV$$

Bring the constants in front and multiply both sides by 2.

$$\iiint_D \frac{\partial}{\partial t}(w^2)\,dV = 2k \iiint_D w\Delta w\,dV$$

Bring the time derivative in front of the integral on the left. Since the integral wipes out the x, y, and z variables, the derivative in front is a total derivative.

$$\frac{d}{dt}\iiint_D w^2\,dV = 2k \iiint_D w\Delta w\,dV \tag{2}$$

Taking the two arbitrary functions to be w, Green's first identity says that

$$\iint_{\text{bdy } D} w\frac{\partial w}{\partial n}\,dS = \iiint_D |\nabla w|^2\,dV + \iiint_D w\Delta w\,dV$$

$$0 = \iiint_D |\nabla w|^2\,dV + \iiint_D w\Delta w\,dV \quad \rightarrow \quad \iiint_D w\Delta w\,dV = -\iiint_D |\nabla w|^2\,dV.$$

As a result, equation (2) becomes

$$\frac{d}{dt}\iiint_D w^2\,dV = -2k \iiint_D |\nabla w|^2\,dV.$$

Integrate both sides with respect to t.

$$\iiint_D w^2\,dV = -2k \int_0^t \iiint_D |\nabla w|^2\,dV\,ds + C$$

The lower limit of integration is arbitrary as long as C is present and has been set to zero. Set $t = 0$ in the equation to determine C.

$$\iiint_D [w(x, y, z, 0)]^2\,dV = C$$

Since $w = 0$ in D at $t = 0$, $C = 0$. Consequently, the previous equation becomes

$$\iiint_D w^2\,dV = -2k \int_0^t \iiint_D |\nabla w|^2\,dV\,ds.$$

Both integrands are positive, so the integrals are as well. Due to the minus sign and the fact that k is positive, both sides must be equal to zero.

$$\iiint_D w^2\,dV = -2k \int_0^t \iiint_D |\nabla w|^2\,dV\,ds = 0$$

This equation is equivalent to these two.

$$\begin{cases} \displaystyle\iiint_D w^2\,dV = 0 \\ \displaystyle\int_0^t \iiint_D |\nabla w|^2\,dV\,ds = 0 \end{cases}$$

By the vanishing theorem, both integrands must be equal to zero.

$$\begin{cases} w^2 = 0 & \text{in } D \\ |\nabla w|^2 = 0 & \text{in } D \end{cases}$$

$$\begin{cases} w = 0 & \text{in } D \\ \nabla w = \mathbf{0} & \text{in } D \end{cases}$$

$$\begin{cases} w = 0 & \text{in } D \\ w = \text{constant} & \text{in } D \end{cases}$$

In order for w to be consistent with its value on the boundary of D, this constant has to be zero.

$$\begin{cases} w = 0 & \text{in } D \\ w = 0 & \text{in } D \end{cases}$$

$w = 0$ implies that $u = v$, which means that the two solutions to the diffusion equation must be one and the same function.