Exercise 5

Prove Dirichlet's principle for the Neumann boundary condition. It asserts that among *all* real-valued functions $w(\mathbf{x})$ on D the quantity

$$E[w] = \frac{1}{2} \iiint_D |\nabla w|^2 \, d\mathbf{x} - \iint_{\text{bdy } D} hw \, dS$$

is the smallest for $w = u$, where u is the solution of the Neumann problem

$$-\Delta u = 0 \quad \text{in } D, \qquad \frac{\partial u}{\partial n} = h(\mathbf{x}) \quad \text{on bdy } D.$$

It is required to assume that the average of the given function $h(\mathbf{x})$ is zero (by Exercise 6.1.11).

Notice three features of this principle:

(i) There is *no constraint at all* on the trial functions $w(\mathbf{x})$.

(ii) The function $h(\mathbf{x})$ appears in the energy.

(iii) The functional $E[w]$ does not change if a constant is added to $w(\mathbf{x})$.

(*Hint:* Follow the method in Section 7.1.)

Solution

Let u satisfy the Neumann problem,

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D \\ \frac{\partial u}{\partial n} &= h \quad \text{on bdy } D \end{aligned}$$

and let w be a real-valued function in D that satisfies $w = u - v$. As a result, the energy becomes

$$\begin{aligned} E[w] &= \frac{1}{2} \iiint_D |\nabla w|^2 \, dV - \iint_{\text{bdy } D} hw \, dS \\ &= \frac{1}{2} \iiint_D \nabla w \cdot \nabla w \, dV - \iint_{\text{bdy } D} hw \, dS \\ &= \frac{1}{2} \iiint_D \nabla(u-v) \cdot \nabla(u-v) \, dV - \iint_{\text{bdy } D} h(u-v) \, dS \\ &= \frac{1}{2} \iiint_D (\nabla u - \nabla v) \cdot (\nabla u - \nabla v) \, dV - \iint_{\text{bdy } D} (hu - hv) \, dS \\ &= \frac{1}{2} \iiint_D (\nabla u \cdot \nabla u - 2\nabla u \cdot \nabla v + \nabla v \cdot \nabla v) \, dV - \iint_{\text{bdy } D} hu \, dS + \iint_{\text{bdy } D} hv \, dS \\ &= \frac{1}{2} \iiint_D (|\nabla u|^2 - 2\nabla u \cdot \nabla v + |\nabla v|^2) \, dV - \iint_{\text{bdy } D} hu \, dS + \iint_{\text{bdy } D} hv \, dS. \end{aligned}$$

Split up the integrals and then use the definition of energy to combine the first and fourth terms.

$$\begin{aligned} E[w] &= \frac{1}{2}\iiint_D |\nabla u|^2\,dV + \frac{1}{2}\iiint_D |\nabla v|^2\,dV - \iiint_D \nabla u\cdot\nabla v\,dV - \iint_{\text{bdy } D} hu\,dS + \iint_{\text{bdy } D} hv\,dS \\ &= E[u] + \frac{1}{2}\iiint_D |\nabla v|^2\,dV - \iiint_D \nabla u\cdot\nabla v\,dV + \iint_{\text{bdy } D} hv\,dS \end{aligned}$$

Green's first identity says that for two arbitrary functions, u and v,

$$\iint_{\text{bdy } D} v\frac{\partial u}{\partial n}\,dS = \iiint_D \nabla u\cdot\nabla v\,dV + \iiint_D v\Delta u\,dV \quad \rightarrow \quad \iiint_D \nabla u\cdot\nabla v\,dV = \iint_{\text{bdy } D} v\frac{\partial u}{\partial n}\,dS - \iiint_D v\Delta u\,dV$$

so the previous equation becomes

$$\begin{aligned} E[w] &= E[u] + \frac{1}{2}\iiint_D |\nabla v|^2\,dV - \iint_{\text{bdy } D} v\frac{\partial u}{\partial n}\,dS + \iiint_D v\Delta u\,dV + \iint_{\text{bdy } D} hv\,dS \\ &= E[u] + \frac{1}{2}\iiint_D |\nabla v|^2\,dV - \iint_{\text{bdy } D} v(h)\,dS + \iiint_D v(0)\,dV + \iint_{\text{bdy } D} hv\,dS \\ &= E[u] + \frac{1}{2}\iiint_D |\nabla v|^2\,dV - \iint_{\text{bdy } D} hv\,dS + \iint_{\text{bdy } D} hv\,dS \\ &= E[u] + \frac{1}{2}\iiint_D |\nabla v|^2\,dV. \end{aligned}$$

The second term is positive, so $E[w] \geq E[u]$. Therefore, the energy E is minimized if $w = u$.