Exercise 6

Let A and B be two disjoint bounded spatial domains, and let D be their exterior. So bdy $D =$ bdy $A \cup$ bdy B. Consider a harmonic function $u(\mathbf{x})$ in D that tends to zero at infinity, which is *constant* on bdy A and *constant* on bdy B, and which satisfies

$$\iint_{\text{bdy } A} \frac{\partial u}{\partial n}\, dS = Q > 0 \qquad \text{and} \qquad \iint_{\text{bdy } B} \frac{\partial u}{\partial n}\, dS = 0.$$

[*Interpretation:* The harmonic function $u(\mathbf{x})$ is the electrostatic potential of two conductors, A and B; Q is the charge on A, while B is uncharged.]

(a) Show that the solution is unique. (*Hint:* Use the Hopf maximum principle.)

(b) Show that $u \geq 0$ in D. [*Hint:* If not, then $u(\mathbf{x})$ has a negative minimum. Use the Hopf principle again.]

(c) Show that $u > 0$ in D.

Solution

A harmonic function is a function that satisfies the Laplace equation. Suppose that there are two solutions, $u = u(x, y, z)$ and $v = (x, y, z)$, for the potential.

$$\Delta u = 0 \text{ in } D \qquad \iint_{\text{bdy } A} \frac{\partial u}{\partial n}\, dS = Q \qquad \iint_{\text{bdy } B} \frac{\partial u}{\partial n}\, dS = 0 \qquad u = \begin{cases} C_1 & \text{on bdy } A \\ C_2 & \text{on bdy } B \end{cases} \qquad \lim_{|\mathbf{x}|\to\infty} u = 0$$

$$\Delta v = 0 \text{ in } D \qquad \iint_{\text{bdy } A} \frac{\partial v}{\partial n}\, dS = Q \qquad \iint_{\text{bdy } B} \frac{\partial v}{\partial n}\, dS = 0 \qquad v = \begin{cases} C_3 & \text{on bdy } A \\ C_4 & \text{on bdy } B \end{cases} \qquad \lim_{|\mathbf{x}|\to\infty} v = 0$$

Subtract both sides of each equation on the bottom from those of the equation above it.

$$\Delta(u - v) = 0 \text{ in } D \qquad \iint_{\text{bdy } A} \frac{\partial}{\partial n}(u - v)\, dS = Q - Q = 0 \qquad \iint_{\text{bdy } B} \frac{\partial}{\partial n}(u - v)\, dS = 0$$

$$u - v = \begin{cases} C_1 - C_3 & \text{on bdy } A \\ C_2 - C_4 & \text{on bdy } B \end{cases} \qquad \lim_{|\mathbf{x}|\to\infty} (u - v) = 0$$

Let w be the difference of u and v: $w = u - v$.

$$\Delta w = 0 \text{ in } D \qquad \iint_{\text{bdy } A} \frac{\partial w}{\partial n}\, dS = 0 \qquad \iint_{\text{bdy } B} \frac{\partial w}{\partial n}\, dS = 0$$

$$w = \begin{cases} C_1 - C_3 & \text{on bdy } A \\ C_2 - C_4 & \text{on bdy } B \end{cases} \qquad \lim_{|\mathbf{x}|\to\infty} w = 0$$

Setting both functions equal to w in Green's first identity gives

$$\iint_{\text{bdy } D} w \frac{\partial w}{\partial n}\, dS = \iiint_D \nabla w \cdot \nabla w\, dV + \iiint_D w \Delta w\, dV.$$

Since $\Delta w = 0$ in D, the second term on the right side is zero.

$$\iint\limits_{\text{bdy } D} w \frac{\partial w}{\partial n}\, dS = \iiint\limits_{D} \nabla w \cdot \nabla w \, dV$$

Use the fact that bdy D = bdy $A \cup$ bdy B.

$$\iint\limits_{\text{bdy } A \,\cup\, \text{bdy } B} w \frac{\partial w}{\partial n}\, dS = \iiint\limits_{D} \nabla w \cdot \nabla w \, dV$$

Split up the integral on the left side.

$$\iint\limits_{\text{bdy } A} w \frac{\partial w}{\partial n}\, dS + \iint\limits_{\text{bdy } B} w \frac{\partial w}{\partial n}\, dS = \iiint\limits_{D} \nabla w \cdot \nabla w \, dV$$

Substitute the value of w on each boundary.

$$\iint\limits_{\text{bdy } A} (C_1 - C_3) \frac{\partial w}{\partial n}\, dS + \iint\limits_{\text{bdy } B} (C_2 - C_4) \frac{\partial w}{\partial n}\, dS = \iiint\limits_{D} \nabla w \cdot \nabla w \, dV$$

Bring the constants in front of the integrals.

$$(C_1 - C_3) \underbrace{\iint\limits_{\text{bdy } A} \frac{\partial w}{\partial n}\, dS}_{=\,0} + (C_2 - C_4) \underbrace{\iint\limits_{\text{bdy } B} \frac{\partial w}{\partial n}\, dS}_{=\,0} = \iiint\limits_{D} \nabla w \cdot \nabla w \, dV$$

As a result,

$$\begin{aligned} 0 &= \iiint\limits_{D} \nabla w \cdot \nabla w \, dV \\ &= \iiint\limits_{D} \langle w_x, w_y, w_z \rangle \cdot \langle w_x, w_y, w_z \rangle \, dV \\ &= \iiint\limits_{D} (w_x^2 + w_y^2 + w_z^2)\, dV. \end{aligned}$$

By the vanishing theorem, the integrand must be zero.

$$w_x^2 + w_y^2 + w_z^2 = 0$$

This implies that w_x, w_y, and w_z are zero individually,

$$\begin{aligned} w_x &= 0 \\ w_y &= 0 \\ w_z &= 0, \end{aligned}$$

which means that w is a constant in D. In order to satisfy the condition at infinity, this constant must be zero.

$$w = 0$$

Therefore, the solution for the potential is unique.

Setting both functions equal to u in Green's first identity gives

$$\iint_{\text{bdy } D} u\frac{\partial u}{\partial n}\,dS = \iiint_D \nabla u \cdot \nabla u\,dV + \iiint_D u\Delta u\,dV.$$

Since $\Delta u = 0$ in D, the second term on the right side is zero.

$$\iint_{\text{bdy } D} u\frac{\partial u}{\partial n}\,dS = \iiint_D \nabla u \cdot \nabla u\,dV$$

Use the fact that bdy $D =$ bdy $A \cup$ bdy B.

$$\iint_{\text{bdy } A \,\cup\, \text{bdy } B} u\frac{\partial u}{\partial n}\,dS = \iiint_D \nabla u \cdot \nabla u\,dV$$

Split up the integral on the left side.

$$\iint_{\text{bdy } A} u\frac{\partial u}{\partial n}\,dS + \iint_{\text{bdy } B} u\frac{\partial u}{\partial n}\,dS = \iiint_D \nabla u \cdot \nabla u\,dV$$

Substitute the value of u on each boundary.

$$\iint_{\text{bdy } A} (C_1)\frac{\partial u}{\partial n}\,dS + \iint_{\text{bdy } B} (C_2)\frac{\partial u}{\partial n}\,dS = \iiint_D \nabla u \cdot \nabla u\,dV$$

Bring the constants in front of the integrals.

$$C_1 \underbrace{\iint_{\text{bdy } A} \frac{\partial u}{\partial n}\,dS}_{=Q} + C_2 \underbrace{\iint_{\text{bdy } B} \frac{\partial u}{\partial n}\,dS}_{=0} = \iiint_D \nabla u \cdot \nabla u\,dV$$

$$C_1 Q = \iiint_D (u_x^2 + u_y^2 + u_z^2)\,dV$$

The potential at the boundary of A is now known.

$$C_1 = \frac{1}{Q}\iiint_D (u_x^2 + u_y^2 + u_z^2)\,dV$$

It's positive because both Q and the integrand are positive. The fact that conductor B is uncharged (or grounded) means that the potential at its surface is zero: $C_2 = 0$.

$$u = \begin{cases} \dfrac{1}{Q}\displaystyle\iiint_D (u_x^2 + u_y^2 + u_z^2)\,dV & \text{on bdy } A \\ 0 & \text{on bdy } B \end{cases}$$

According to the maximum principle for the Laplace equation, the maximum and minimum of u can only be attained on bdy A or bdy B. Therefore,

$$0 < u < \frac{1}{Q}\iiint_D (u_x^2 + u_y^2 + u_z^2)\,dV \quad \text{in } D.$$