Exercise 9

Repeat Exercise 8 with the same choice of w_0 and w_1 and with $w_2 = x^2y(3-3x-y)$. That is, find the Rayleigh–Ritz approximation $w_0 + c_1w_1 + c_2w_2$ to u.

Solution

The quantity $w_0 + c_1w_1 + c_2w_2$ satisfies the Laplace equation. Recall that $w_0 = y(3-3x-y)$, $w_1 = xy(3-3x-y)$, and D is the triangular region $\{x > 0,\ y > 0,\ 3x + y < 3\}$.

$$\Delta(w_0 + c_1w_1 + c_2w_2) = 0$$

The Laplacian operator $\Delta = \partial_x^2 + \partial_y^2$ is linear.

$$\Delta w_0 + c_1\Delta w_1 + c_2\Delta w_2 = 0$$

Multiply both sides by w_j, where j is an integer $1 \le j \le 2$.

$$w_j\Delta w_0 + c_1w_j\Delta w_1 + c_2w_j\Delta w_2 = 0$$

Integrate both sides over the area D of the triangle in the xy-plane.

$$\iint_D w_j\Delta w_0\,dA + c_1\iint_D w_j\Delta w_1\,dA + c_2\iint_D w_j\Delta w_2\,dA = 0 \tag{1}$$

The analog of Green's first identity in two dimensions is

$$\int_{\text{bdy } D} v\frac{\partial u}{\partial n}\,ds = \iint_D \nabla v\cdot\nabla u\,dA + \iint_D v\Delta u\,dA.$$

Let $u = w_i$ and $v = w_j$, where i is an integer $0 \le i \le 2$.

$$\int_{\text{bdy } D} w_j\frac{\partial w_i}{\partial n}\,ds = \iint_D \nabla w_j\cdot\nabla w_i\,dA + \iint_D w_j\Delta w_i\,dA$$

Since $w_j = 0$ on the boundary of D, the line integral on the left is zero.

$$0 = \iint_D \nabla w_j\cdot\nabla w_i\,dA + \iint_D w_j\Delta w_i\,dA$$

Consequently,

$$\iint_D w_j\Delta w_i\,dA = -\iint_D \nabla w_j\cdot\nabla w_i\,dA,$$

and equation (1) becomes

$$-\iint_D \nabla w_j\cdot\nabla w_0\,dA - c_1\iint_D \nabla w_j\cdot\nabla w_1\,dA - c_2\iint_D \nabla w_j\cdot\nabla w_2\,dA = 0.$$

The system of equations for c_1 and c_2 is thus

$$-\iint_D \nabla w_1 \cdot \nabla w_0 \, dA - c_1 \iint_D \nabla w_1 \cdot \nabla w_1 \, dA - c_2 \iint_D \nabla w_1 \cdot \nabla w_2 \, dA = 0$$
$$-\iint_D \nabla w_2 \cdot \nabla w_0 \, dA - c_1 \iint_D \nabla w_2 \cdot \nabla w_1 \, dA - c_2 \iint_D \nabla w_2 \cdot \nabla w_2 \, dA = 0.$$

Evaluate the six double integrals.

$$\begin{aligned}\iint_D \nabla w_1 \cdot \nabla w_0 \, dA &= \int_0^1 \int_0^{3-3x} \langle -y(-3+6x+y), x(3-3x-2y)\rangle \cdot \langle -3y, 3-3x-2y\rangle \, dy\, dx \\ &= \int_0^1 \int_0^{3-3x} [3y^2(-3+6x+y) + x(3-3x-2y)^2] \, dy\, dx \\ &= 0.45\end{aligned}$$

$$\begin{aligned}\iint_D \nabla w_1 \cdot \nabla w_1 \, dA &= \int_0^1 \int_0^{3-3x} \langle -y(-3+6x+y), x(3-3x-2y)\rangle \cdot \langle -y(-3+6x+y), x(3-3x-2y)\rangle \, dy\, dx \\ &= \int_0^1 \int_0^{3-3x} [y^2(-3+6x+y)^2 + x^2(3-3x-2y)^2] \, dy\, dx \\ &= 1.5\end{aligned}$$

$$\begin{aligned}\iint_D \nabla w_1 \cdot \nabla w_2 \, dA &= \int_0^1 \int_0^{3-3x} \langle -y(-3+6x+y), x(3-3x-2y)\rangle \cdot \langle xy(6-9x-2y), x^2(3-3x-2y)\rangle \, dy\, dx \\ &= \int_0^1 \int_0^{3-3x} [-xy^2(-3+6x+y)(6-9x-2y) + x^3(3-3x-2y)^2] \, dy\, dx \\ &= 0.45\end{aligned}$$

$$\begin{aligned}\iint_D \nabla w_2 \cdot \nabla w_0 \, dA &= \int_0^1 \int_0^{3-3x} \langle xy(6-9x-2y), x^2(3-3x-2y)\rangle \cdot \langle -3y, 3-3x-2y\rangle \, dy\, dx \\ &= \int_0^1 \int_0^{3-3x} [-3xy^2(6-9x-2y) + x^2(3-3x-2y)^2] \, dy\, dx \\ &= 0.15\end{aligned}$$

$$\begin{aligned}\iint_D \nabla w_2 \cdot \nabla w_1 \, dA &= \iint_D \nabla w_1 \cdot \nabla w_2 \, dA \\ &= 0.45\end{aligned}$$

$$\begin{aligned}\iint_D \nabla w_2 \cdot \nabla w_2 \, dA &= \int_0^1 \int_0^{3-3x} \langle xy(6-9x-2y), x^2(3-3x-2y)\rangle \cdot \langle xy(6-9x-2y), x^2(3-3x-2y)\rangle \, dy\, dx \\ &= \int_0^1 \int_0^{3-3x} [x^2y^2(6-9x-2y)^2 + x^4(3-3x-2y)^2] \, dy\, dx \\ &= 0.225\end{aligned}$$

Substituting these results into the system, it becomes

$$-0.45 - 1.5c_1 - 0.45c_2 = 0$$
$$-0.15 - 0.45c_1 - 0.225c_2 = 0,$$

which yields $c_1 = -1/4$ and $c_2 = -1/6$. Therefore, the Rayleigh–Ritz approximation $w_0 + c_1 w_1 + c_2 w_2$ to u is

$$u \approx y(3 - 3x - y) - \frac{1}{4}xy(3 - 3x - y) - \frac{1}{6}x^2 y(3 - 3x - y).$$

This answer is in disagreement with the answer at the back of the book, $c_1 = -0.248$ and $c_2 = -0.008$.