Exercise 10

Let $u(x, y)$ be the harmonic function in the unit disk with the boundary values $u(x, y) = x^2$ on $\{x^2 + y^2 = 1\}$. Find its Rayleigh–Ritz approximation of the form $x^2 + c_1(1 - x^2 - y^2)$.

Solution

The Hard Way

The quantity $w_0 + c_1 w_1 = x^2 + c_1(1 - x^2 - y^2)$ satisfies the Laplace equation.

$$\Delta(w_0 + c_1 w_1) = 0$$

The Laplacian operator $\Delta = \partial_x^2 + \partial_y^2$ is linear.

$$\Delta w_0 + c_1 \Delta w_1 = 0$$

Multiply both sides by w_1.

$$w_1 \Delta w_0 + c_1 w_1 \Delta w_1 = 0$$

Integrate both sides over the area D of the triangle in the xy-plane.

$$\iint_D w_1 \Delta w_0 \, dA + c_1 \iint_D w_1 \Delta w_1 \, dA = 0 \tag{1}$$

The analog of Green's first identity in two dimensions is

$$\int_{\text{bdy } D} v \frac{\partial u}{\partial n} \, ds = \iint_D \nabla v \cdot \nabla u \, dA + \iiint_D v \Delta u \, dA.$$

Let $u = w_i$ and $v = w_1$, where i is an integer $0 \le i \le 1$.

$$\int_{\text{bdy } D} w_1 \frac{\partial w_i}{\partial n} \, ds = \iint_D \nabla w_1 \cdot \nabla w_i \, dA + \iint_D w_1 \Delta w_i \, dA$$

Since $w_1 = 0$ on the boundary of D, the line integral on the left is zero.

$$0 = \iint_D \nabla w_1 \cdot \nabla w_i \, dA + \iint_D w_1 \Delta w_i \, dA$$

Consequently,

$$\iint_D w_1 \Delta w_i \, dA = -\iint_D \nabla w_1 \cdot \nabla w_i \, dA,$$

and equation (1) becomes

$$-\iint_D \nabla w_1 \cdot \nabla w_0 \, dA - c_1 \iint_D \nabla w_1 \cdot \nabla w_1 \, dA = 0.$$

Solve it for c_1 and evaluate the double integrals.

$$\begin{aligned}
c_1 &= -\frac{\iint_D \nabla w_1 \cdot \nabla w_0 \, dA}{\iint_D \nabla w_1 \cdot \nabla w_1 \, dA} \\
&= -\frac{\iint\limits_{x^2+y^2\leq 1} \langle -2x, -2y\rangle \cdot \langle 2x, 0\rangle \, dA}{\iint\limits_{x^2+y^2\leq 1} \langle -2x, -2y\rangle \cdot \langle -2x, -2y\rangle \, dA} \\
&= -\frac{\iint\limits_{x^2+y^2\leq 1} (-4x^2) \, dA}{\iint\limits_{x^2+y^2\leq 1} (4x^2+4y^2) \, dA} \\
&= \frac{\iint\limits_{x^2+y^2\leq 1} x^2 \, dA}{\iint\limits_{x^2+y^2\leq 1} (x^2+y^2) \, dA} \\
&= \frac{\int_0^{2\pi}\int_0^1 r^2 \cos^2\theta \, (r\, dr\, d\theta)}{\int_0^{2\pi}\int_0^1 r^2 \, (r\, dr\, d\theta)} = \frac{\pi/4}{\pi/2} = \frac{1}{2}
\end{aligned}$$

Therefore, the Rayleigh–Ritz approximation $w_0 + c_1 w_1$ to u is

$$u \approx x^2 + \frac{1}{2}(1 - x^2 - y^2).$$

The Easy Way

Since there's only one constant to determine, the Rayleigh–Ritz approximation can be substituted directly into the Laplace equation.

$$\begin{gathered}
u_{xx} + u_{yy} = 0 \\
\frac{\partial^2}{\partial x^2}[x^2 + c_1(1 - x^2 - y^2)] + \frac{\partial^2}{\partial y^2}[x^2 + c_1(1 - x^2 - y^2)] = 0 \\
(2 - 2c_1) + (-2c_1) = 0 \\
-4c_1 = -2 \\
c_1 = \frac{1}{2}
\end{gathered}$$

Therefore, the Rayleigh–Ritz approximation $w_0 + c_1 w_1$ to u is

$$u \approx x^2 + \frac{1}{2}(1 - x^2 - y^2).$$