Exercise 1

Derive the representation fomula for harmonic functions (7.2.5) in two dimensions.

Solution

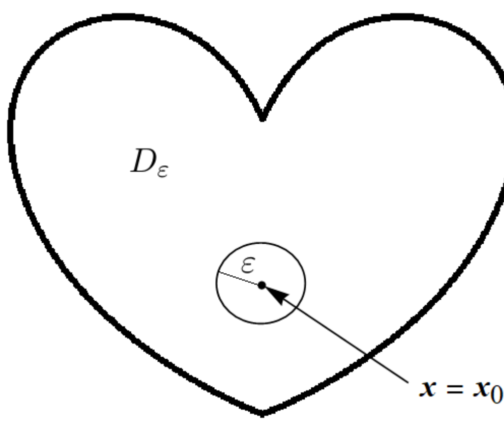
Start with the two-dimensional analog of Green's second identity, which holds for any two functions, $u = u(x, y)$ and $v = v(x, y)$, defined in some domain D.

$$\iint\limits_D (u\Delta v - v\Delta u)\, dA = \int\limits_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds$$

Let u be a harmonic function ($\Delta u = 0$) in D, and let

$$v = \frac{1}{2\pi} \ln r \quad \text{in } D,$$

where $r = |\mathbf{x} - \mathbf{x}_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$. Note that v blows up at $\mathbf{x} = \mathbf{x}_0$. Thus, rather than D, we consider the domain D_ε, which is D with a circular hole of radius ε centered at $\mathbf{x}_0$.



In D_ε, $\Delta u = 0$ and $\Delta v = 0$, so the left side of Green's second identity vanishes.

$$\iint\limits_{D_\varepsilon} (u\Delta v - v\Delta u)\, dA = \int\limits_{\text{bdy } D_\varepsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds$$

$$\iint\limits_{D_\varepsilon} [u(0) - v(0)]\, dA = \int\limits_{\text{bdy } D_\varepsilon} \left[u\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\ln r\right) - \left(\frac{1}{2\pi}\ln r\right)\frac{\partial u}{\partial n}\right] ds$$

$$0 = \frac{1}{2\pi}\int\limits_{\text{bdy } D_\varepsilon} \left[u\frac{\partial}{\partial n}(\ln r) - \frac{\partial u}{\partial n}\ln r\right] ds$$

D_ε has two boundaries, the circle and the heart—the latter is the boundary of D.

$$0 = \frac{1}{2\pi}\int\limits_{\text{bdy } D} \left[u\frac{\partial}{\partial n}(\ln r) - \frac{\partial u}{\partial n}\ln r\right] ds + \frac{1}{2\pi}\int\limits_{\substack{(x-x_0)^2\\+(y-y_0)^2=\varepsilon^2}} \left[u\frac{\partial}{\partial n}(\ln r) - \frac{\partial u}{\partial n}\ln r\right] ds$$

Bring the first integral to the left side.

$$-\frac{1}{2\pi}\int\limits_{\text{bdy } D}\left[u(\mathbf{x})\frac{\partial}{\partial n}(\ln \mathscr{r}) - \frac{\partial u}{\partial n}\ln \mathscr{r}\right]ds = \frac{1}{2\pi}\int\limits_{\substack{(x-x_0)^2\\+(y-y_0)^2=\varepsilon^2}}\left[u(\mathbf{x})\frac{\partial}{\partial n}(\ln \mathscr{r}) - \frac{\partial u}{\partial n}\ln \mathscr{r}\right]ds$$

Along the circle the normal derivative points toward the center: $\partial/\partial n = -\partial/\partial \mathscr{r}$.

$$\begin{aligned}
-\frac{1}{2\pi}\int\limits_{\text{bdy } D}\left[u(\mathbf{x})\frac{\partial}{\partial n}(\ln \mathscr{r}) - \frac{\partial u}{\partial n}\ln \mathscr{r}\right]ds &= \frac{1}{2\pi}\int\limits_{\mathscr{r}=\varepsilon}\left[-u(\mathbf{x})\frac{\partial}{\partial \mathscr{r}}(\ln \mathscr{r}) + \frac{\partial u}{\partial \mathscr{r}}\ln \mathscr{r}\right]ds \\
&= \frac{1}{2\pi}\int\limits_{\mathscr{r}=\varepsilon}\left[-u(\mathbf{x})\left(\frac{1}{\mathscr{r}}\right) + \frac{\partial u}{\partial \mathscr{r}}\ln \mathscr{r}\right]ds \\
&= \frac{1}{2\pi}\int\limits_{\mathscr{r}=\varepsilon}\left[-u(\mathbf{x})\left(\frac{1}{\varepsilon}\right) + \frac{\partial u}{\partial \mathscr{r}}\ln \varepsilon\right]ds \\
&= -\frac{1}{2\pi}\int\limits_{\mathscr{r}=\varepsilon}u(\mathbf{x})\left(\frac{1}{\varepsilon}\right)ds + \frac{1}{2\pi}\int_{\mathscr{r}=\varepsilon}\frac{\partial u}{\partial \mathscr{r}}\ln \varepsilon\, ds \\
&= -\frac{1}{2\pi\varepsilon}\int\limits_{\mathscr{r}=\varepsilon}u(\mathbf{x})\,ds + \frac{\varepsilon\ln\varepsilon}{2\pi\varepsilon}\int\limits_{\mathscr{r}=\varepsilon}\frac{\partial u}{\partial \mathscr{r}}\,ds \\
&= -\frac{\int_{\mathscr{r}=\varepsilon}u(\mathbf{x})\,ds}{\int_{\mathscr{r}=\varepsilon}ds} + \varepsilon\ln\varepsilon\,\frac{\int_{\mathscr{r}=\varepsilon}\frac{\partial u}{\partial \mathscr{r}}\,ds}{\int_{\mathscr{r}=\varepsilon}ds} \\
&= -\bar{u} + \varepsilon\ln\varepsilon\,\overline{\frac{\partial u}{\partial \mathscr{r}}}
\end{aligned}$$

The overbar here represents the average of the quantity below it over the circle of radius ε centered at $\mathbf{x}_0$. Take the limit now as $\varepsilon \to 0$. Since

$$\lim_{\varepsilon\to 0}\varepsilon\ln\varepsilon = \lim_{\varepsilon\to 0}\frac{\ln\varepsilon}{\frac{1}{\varepsilon}} \overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \lim_{\varepsilon\to 0}\frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = \lim_{\varepsilon\to 0}(-\varepsilon) = 0,$$

the second term on the right vanishes. As $\varepsilon \to 0$, $\bar{u}$ tends to the value of $u(\mathbf{x}_0)$.

$$-\frac{1}{2\pi}\int\limits_{\text{bdy } D}\left[u(\mathbf{x})\frac{\partial}{\partial n}(\ln \mathscr{r}) - \frac{\partial u}{\partial n}\ln \mathscr{r}\right]ds = -u(\mathbf{x}_0)$$

Therefore, the representation formula in two dimensions is

$$u(\mathbf{x}_0) = \frac{1}{2\pi}\int\limits_{\text{bdy } D}\left[u(\mathbf{x})\frac{\partial}{\partial n}(\ln|\mathbf{x}-\mathbf{x}_0|) - \frac{\partial u}{\partial n}\ln|\mathbf{x}-\mathbf{x}_0|\right]ds.$$