# Exercise 2

Let $\phi(\mathbf{x})$ be any $C^2$ function defined on all of three-dimensional space that vanishes outside some sphere. Show that

$$\phi(\mathbf{0}) = -\iiint \frac{1}{|\mathbf{x}|}\Delta\phi(\mathbf{x})\frac{d\mathbf{x}}{4\pi}.$$

The integration is taken over the region where $\phi(\mathbf{x})$ is not zero.

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### Solution

Start with Green's second identity, which holds for any two functions, $u = u(x, y, z)$ and $v = v(x, y, z)$, defined in some domain $D$.

$$\iiint_D (u\Delta v - v\Delta u)\, dV = \iint_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$$

Let $u = \phi$ in $D$, and let

$$v = -\frac{1}{4\pi r} \quad \text{in } D,$$

where $r = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$. Here we choose $D$ to be the set of points exterior to the sphere of radius $R$ centered at the origin. As a result, $\Delta v = 0$ in $D$ and Green's second identity becomes

$$\iiint_D (\phi\Delta v - v\Delta\phi)\, dV = \iint_{\text{bdy } D} \left(\phi\frac{\partial v}{\partial n} - v\frac{\partial \phi}{\partial n}\right) dS$$

$$\iiint_D \left[\phi(\mathbf{x})(0) - \left(-\frac{1}{4\pi r}\right)\Delta\phi(\mathbf{x})\right] dV = \iint_{\text{bdy } D} \left[\phi(\mathbf{x})\frac{\partial}{\partial n}\left(-\frac{1}{4\pi r}\right) - \left(-\frac{1}{4\pi r}\right)\frac{\partial\phi}{\partial n}\right] dS$$

$$\iiint_D \frac{\Delta\phi(\mathbf{x})}{4\pi r}\, dV = \frac{1}{4\pi}\iint_{\text{bdy } D} \left[-\phi(\mathbf{x})\frac{\partial}{\partial n}\left(\frac{1}{r}\right) + \left(\frac{1}{r}\right)\frac{\partial\phi}{\partial n}\right] dS.$$

On the sphere the normal derivative points toward the center: $\partial/\partial n = -\partial/\partial r$.

$$\begin{aligned}
\iiint_D \frac{\Delta\phi(\mathbf{x})}{4\pi r}\, dV &= \frac{1}{4\pi}\iint_{x^2+y^2+z^2=R^2} \left[\phi(\mathbf{x})\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \left(\frac{1}{r}\right)\frac{\partial\phi}{\partial r}\right] dS \\
&= \frac{1}{4\pi}\iint_{r=R} \left[-\phi(\mathbf{x})\left(\frac{1}{r^2}\right) - \left(\frac{1}{r}\right)\frac{\partial\phi}{\partial r}\right] dS \\
&= \frac{1}{4\pi}\iint_{r=R} \left[-\phi(\mathbf{x})\left(\frac{1}{R^2}\right) - \left(\frac{1}{R}\right)\frac{\partial\phi}{\partial r}\right] dS \\
&= -\frac{1}{4\pi}\iint_{r=R} \phi(\mathbf{x})\left(\frac{1}{R^2}\right) dS - \frac{1}{4\pi}\iint_{r=R} \left(\frac{1}{R}\right)\frac{\partial\phi}{\partial r}\, dS \\
&= -\frac{1}{4\pi R^2}\iint_{r=R} \phi(\mathbf{x})\, dS - \frac{R}{4\pi R^2}\iint_{r=R} \frac{\partial\phi}{\partial r}\, dS
\end{aligned}$$

Continue the simplification of the right side.

$$\iiint_D \frac{\Delta\phi(\mathbf{x})}{4\pi r}\,dV = -\frac{\iint_{r=R} \phi(\mathbf{x})\,dS}{\iint_{r=R} dS} - R\,\frac{\iint_{r=R} \frac{\partial\phi}{\partial r}\,dS}{\iint_{r=R} dS}$$

$$= -\overline{\phi} - R\overline{\frac{\partial\phi}{\partial r}}$$

The overbar here represents the average of the quantity below it over the sphere of radius $R$ centered at the origin. Taking the limit now as $R \to 0$, the second term on the right vanishes. As $R \to 0$, $\overline{\phi}$ tends to $\phi(\mathbf{0})$, the value of $\phi$ at the origin.

$$\iiint_D \frac{\Delta\phi(\mathbf{x})}{4\pi r}\,dV = -\phi(\mathbf{0})$$

Therefore,

$$\phi(\mathbf{0}) = -\iiint \frac{1}{|\mathbf{x}|}\Delta\phi(\mathbf{x})\frac{d\mathbf{x}}{4\pi},$$

where the integration is taken over the region where $\phi(\mathbf{x})$ is not zero.