Exercise 2

Prove Theorem 2, which gives the solution of Poisson's equation in terms of the Green's function.

Solution

Consider two arbitrary functions, $u = u(x, y, z)$ and $v = v(x, y, z)$, that are defined in some domain of space D. By the chain rule, we have

$$\frac{\partial}{\partial x}(vu_x) = v_x u_x + vu_{xx}$$
$$\frac{\partial}{\partial y}(vu_y) = v_y u_y + vu_{yy}$$
$$\frac{\partial}{\partial z}(vu_z) = v_z u_z + vu_{zz}.$$

Add the respective sides of each equation.

$$\frac{\partial}{\partial x}(vu_x) + \frac{\partial}{\partial y}(vu_y) + \frac{\partial}{\partial z}(vu_z) = (v_x u_x + v_y u_y + v_z u_z) + v(u_{xx} + u_{yy} + u_{zz})$$

Express both sides in terms of familiar vector operators.

$$\nabla \cdot (v\nabla u) = \nabla v \cdot \nabla u + v\Delta u$$

Integrate both sides over the volume of D.

$$\iiint_D \nabla \cdot (v\nabla u)\, dV = \iiint_D (\nabla v \cdot \nabla u + v\Delta u)\, dV$$

Apply the divergence theorem on the left side to turn the volume integral into a surface integral over the boundary of D, using $\mathbf{n}$ to represent the unit vector normal to this boundary.

$$\iint_{\text{bdy } D} v\nabla u \cdot \mathbf{n}\, dS = \iiint_D (\nabla v \cdot \nabla u + v\Delta u)\, dV$$

Use the notation $\partial u/\partial n$ for $\nabla u \cdot \mathbf{n}$ and split up the integral on the right side.

$$\iint_{\text{bdy } D} v\frac{\partial u}{\partial n}\, dS = \iiint_D \nabla v \cdot \nabla u\, dV + \iiint_D v\Delta u\, dV \tag{1}$$

This is Green's first identity. Since u and v are arbitrary, u can be replaced with v, and v can be replaced with u.

$$\iint_{\text{bdy } D} u\frac{\partial v}{\partial n}\, dS = \iiint_D \nabla u \cdot \nabla v\, dV + \iiint_D u\Delta v\, dV$$

Subtract the respective sides of this equation with those of equation (1).

$$\iint_{\text{bdy } D} u\frac{\partial v}{\partial n}\, dS - \iint_{\text{bdy } D} v\frac{\partial u}{\partial n}\, dS = \iiint_D u\Delta v\, dV - \iiint_D v\Delta u\, dV$$

Combining the surface and volume integrals results in Green's second identity.

$$\iint_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS = \iiint_D (u\Delta v - v\Delta u)\, dV \tag{2}$$

Remarkably, the solution to Poisson's equation in three dimensions follows from this identity. Let u satisfy the Dirichlet boundary value problem,

$$\begin{aligned} \Delta u &= f && \text{in } D \\ u &= h && \text{on bdy } D, \end{aligned}$$

and let v be the corresponding Green's function.

$$\begin{aligned} \Delta G(\mathbf{x};\mathbf{x}_0) &= \delta(\mathbf{x}-\mathbf{x}_0) && \text{in } D \\ G &= 0 && \text{on bdy } D \end{aligned}$$

As a result, Green's second identity becomes

$$\begin{aligned} \iint_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \right) dS &= \iiint_D (u\Delta G - G\Delta u)\, dV \\ \iint_{\text{bdy } D} \left[h(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x};\mathbf{x}_0) - (0)\frac{\partial u}{\partial n}(\mathbf{x}) \right] dS &= \iiint_D [u(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0) - G(\mathbf{x};\mathbf{x}_0)f(\mathbf{x})]\, dV \\ \iint_{\text{bdy } D} h(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x};\mathbf{x}_0)\, dS &= \iiint_D u(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0)\, dV - \iiint_D G(\mathbf{x};\mathbf{x}_0)f(\mathbf{x})\, dV \\ \iint_{\text{bdy } D} h(\mathbf{x})\nabla G(\mathbf{x};\mathbf{x}_0)\cdot\mathbf{n}\, dS &= u(\mathbf{x}_0) - \iiint_D G(\mathbf{x};\mathbf{x}_0)f(\mathbf{x})\, dV. \end{aligned}$$

Solve this equation for $u(\mathbf{x}_0)$.

$$u(\mathbf{x}_0) = \iint_{\text{bdy } D} h(\mathbf{x})\nabla G(\mathbf{x};\mathbf{x}_0)\cdot\mathbf{n}\, dS + \iiint_D G(\mathbf{x};\mathbf{x}_0)f(\mathbf{x})\, dV$$

Switch the roles of $\mathbf{x}$ and $\mathbf{x}_0$; x_0, y_0, and z_0 are now the dummy variables of integration, and the gradient operator now applies to x_0, y_0, and z_0.

$$u(\mathbf{x}) = \iint_{\text{bdy } D} h(\mathbf{x}_0)\nabla_0 G(\mathbf{x}_0;\mathbf{x})\cdot\mathbf{n}_0\, dS_0 + \iiint_D G(\mathbf{x}_0;\mathbf{x})f(\mathbf{x}_0)\, dV_0$$

Because the Green's function is symmetric,

$$u(\mathbf{x}) = \iint_{\text{bdy } D} h(\mathbf{x}_0)\nabla_0 G(\mathbf{x};\mathbf{x}_0)\cdot\mathbf{n}_0\, dS_0 + \iiint_D G(\mathbf{x};\mathbf{x}_0)f(\mathbf{x}_0)\, dV_0.$$

Therefore,

$$u(x,y,z) = \iint_{\text{bdy } D} h(x_0,y_0,z_0)\nabla_0 G(x,y,z;x_0,y_0,z_0)\cdot\mathbf{n}_0\, dS_0 + \iiint_D G(x,y,z;x_0,y_0,z_0)f(x_0,y_0,z_0)\, dV_0.$$