Exercise 3

Verify the limit of A_ϵ as claimed in the proof of the symmetry of the Green's function.

Solution

The Limit of A_ϵ

A_ϵ is defined to be the surface integral of $u\,\partial v/\partial n - v\,\partial u/\partial n$ over the sphere of radius ϵ centered at $\mathbf{x} = \mathbf{a}$.

$$A_\epsilon = \iint\limits_{|\mathbf{x}-\mathbf{a}|=\epsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS \tag{1}$$

Also, u and v are defined to be

$$\begin{aligned} u(\mathbf{x}) &= G(\mathbf{x};\mathbf{a}) \qquad\qquad \text{and} \qquad v(\mathbf{x}) = G(\mathbf{x};\mathbf{b}), \\ &= -\frac{1}{4\pi|\mathbf{x}-\mathbf{a}|} + H(\mathbf{x}) \end{aligned}$$

so equation (1) becomes

$$A_\epsilon = \iint\limits_{|\mathbf{x}-\mathbf{a}|=\epsilon} \left[\left(-\frac{1}{4\pi|\mathbf{x}-\mathbf{a}|} + H(\mathbf{x})\right)\frac{\partial v}{\partial n} - v(\mathbf{x})\frac{\partial}{\partial n}\left(-\frac{1}{4\pi|\mathbf{x}-\mathbf{a}|} + H(\mathbf{x})\right)\right] dS.$$

Here we use the notation $r = |\mathbf{x}-\mathbf{a}|$. Then the normal derivative points radially inward towards the sphere's center: $\partial/\partial n = -\partial/\partial r$.

$$\begin{aligned}
A_\epsilon &= \iint\limits_{r=\epsilon} \left[\left(-\frac{1}{4\pi r} + H(\mathbf{x})\right)\left(-\frac{\partial v}{\partial r}\right) + v(\mathbf{x})\frac{\partial}{\partial r}\left(-\frac{1}{4\pi r} + H(\mathbf{x})\right)\right] dS \\
&= \iint\limits_{r=\epsilon} \left[\left(\frac{1}{4\pi r} - H(\mathbf{x})\right)\frac{\partial v}{\partial r} + v(\mathbf{x})\left(\frac{1}{4\pi r^2} + \frac{\partial H}{\partial r}\right)\right] dS \\
&= \iint\limits_{r=\epsilon} \left[\left(\frac{1}{4\pi \epsilon} - H(\mathbf{x})\right)\frac{\partial v}{\partial r} + v(\mathbf{x})\left(\frac{1}{4\pi \epsilon^2} + \frac{\partial H}{\partial r}\right)\right] dS \\
&= \iint\limits_{r=\epsilon} \left[\frac{1}{4\pi\epsilon}\frac{\partial v}{\partial r} - H(\mathbf{x})\frac{\partial v}{\partial r} + \frac{v(\mathbf{x})}{4\pi\epsilon^2} + v(\mathbf{x})\frac{\partial H}{\partial r}\right] dS \\
&= \iint\limits_{r=\epsilon} \frac{1}{4\pi\epsilon}\frac{\partial v}{\partial r}\, dS - \iint\limits_{r=\epsilon} H(\mathbf{x})\frac{\partial v}{\partial r}\, dS + \iint\limits_{r=\epsilon} \frac{v(\mathbf{x})}{4\pi\epsilon^2}\, dS + \iint\limits_{r=\epsilon} v(\mathbf{x})\frac{\partial H}{\partial r}\, dS \\
&= \int_0^\pi \int_0^{2\pi} \frac{1}{4\pi\epsilon}\frac{\partial v}{\partial r}\,(\epsilon^2 \sin\phi\, d\theta\, d\phi) - \int_0^\pi \int_0^{2\pi} H(\mathbf{x})\frac{\partial v}{\partial r}\,(\epsilon^2 \sin\phi\, d\theta\, d\phi) \\
&\qquad + \int_0^\pi \int_0^{2\pi} \frac{v(\mathbf{x})}{4\pi\epsilon^2}\,(\epsilon^2 \sin\phi\, d\theta\, d\phi) + \int_0^\pi \int_0^{2\pi} v(\mathbf{x})\frac{\partial H}{\partial r}\,(\epsilon^2 \sin\phi\, d\theta\, d\phi) \\
&= \frac{\epsilon}{4\pi}\int_0^\pi \int_0^{2\pi} \frac{\partial v}{\partial r}\sin\phi\, d\theta\, d\phi - \epsilon^2\int_0^\pi \int_0^{2\pi} H(\mathbf{x})\frac{\partial v}{\partial r}\sin\phi\, d\theta\, d\phi \\
&\qquad + \frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} v(\mathbf{x})\sin\phi\, d\theta\, d\phi + \epsilon^2\int_0^\pi \int_0^{2\pi} v(\mathbf{x})\frac{\partial H}{\partial r}\sin\phi\, d\theta\, d\phi
\end{aligned}$$

Take the limit of both sides as $\epsilon \to 0$. As a result, all terms vanish except the third.

$$\begin{aligned}\lim_{\epsilon\to 0} A_\epsilon &= \lim_{\epsilon\to 0}\bigg[\frac{\epsilon}{4\pi}\int_0^\pi\int_0^{2\pi}\frac{\partial v}{\partial r}\sin\phi\, d\theta\, d\phi - \epsilon^2\int_0^\pi\int_0^{2\pi} H(\mathbf{x})\frac{\partial v}{\partial r}\sin\phi\, d\theta\, d\phi \\ &\qquad + \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} v(\mathbf{x})\sin\phi\, d\theta\, d\phi + \epsilon^2\int_0^\pi\int_0^{2\pi} v(\mathbf{x})\frac{\partial H}{\partial r}\sin\phi\, d\theta\, d\phi\bigg] \\ &= \lim_{\epsilon\to 0}\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} v(\mathbf{x})\sin\phi\, d\theta\, d\phi\end{aligned}$$

The value of $v(\mathbf{x})$ tends to $v(\mathbf{a})$, a constant, as the radius of the sphere $|\mathbf{x}-\mathbf{a}| = \epsilon$ tends to zero.

$$\begin{aligned}\lim_{\epsilon\to 0} A_\epsilon &= \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} v(\mathbf{a})\sin\phi\, d\theta\, d\phi \\ &= \frac{v(\mathbf{a})}{4\pi}\left(\int_0^\pi \sin\phi\, d\phi\right)\left(\int_0^{2\pi} d\theta\right) \\ &= \frac{v(\mathbf{a})}{4\pi}(2)(2\pi) \\ &= v(\mathbf{a}) \\ &= G(\mathbf{a};\mathbf{b})\end{aligned}$$

The Limit of B_ϵ

B_ϵ is defined to be the surface integral of $u\,\partial v/\partial n - v\,\partial u/\partial n$ over the sphere of radius ϵ centered at $\mathbf{x} = \mathbf{b}$.

$$B_\epsilon = \iint\limits_{|\mathbf{x}-\mathbf{b}|=\epsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS \tag{2}$$

Also, u and v are defined to be

$$u(\mathbf{x}) = G(\mathbf{x};\mathbf{a}) \qquad \text{and} \qquad \begin{aligned} v(\mathbf{x}) &= G(\mathbf{x};\mathbf{b}), \\ &= -\frac{1}{4\pi|\mathbf{x}-\mathbf{b}|} + H(\mathbf{x})\end{aligned}$$

so equation (2) becomes

$$B_\epsilon = \iint\limits_{|\mathbf{x}-\mathbf{b}|=\epsilon}\left[u(\mathbf{x})\frac{\partial}{\partial n}\left(-\frac{1}{4\pi|\mathbf{x}-\mathbf{b}|} + H(\mathbf{x})\right) - \left(-\frac{1}{4\pi|\mathbf{x}-\mathbf{b}|} + H(\mathbf{x})\right)\frac{\partial u}{\partial n}\right] dS.$$

Here we use the notation $s = |\mathbf{x}-\mathbf{b}|$. Then the normal derivative points radially inward towards the sphere's center: $\partial/\partial n = -\partial/\partial s$.

$$\begin{aligned}B_\epsilon &= \iint\limits_{s=\epsilon}\left[-u(\mathbf{x})\frac{\partial}{\partial s}\left(-\frac{1}{4\pi s} + H(\mathbf{x})\right) - \left(-\frac{1}{4\pi s} + H(\mathbf{x})\right)\left(-\frac{\partial u}{\partial s}\right)\right] dS \\ &= \iint\limits_{s=\epsilon}\left[-u(\mathbf{x})\left(\frac{1}{4\pi s^2} + \frac{\partial H}{\partial s}\right) + \left(-\frac{1}{4\pi s} + H(\mathbf{x})\right)\frac{\partial u}{\partial s}\right] dS\end{aligned}$$

On the sphere $s = \epsilon$.

$$\begin{aligned}
B_\epsilon &= \iint_{s=\epsilon} \left[-u(\mathbf{x})\left(\frac{1}{4\pi\epsilon^2} + \frac{\partial H}{\partial s}\right) + \left(-\frac{1}{4\pi\epsilon} + H(\mathbf{x})\right)\frac{\partial u}{\partial s}\right] dS \\
&= \iint_{s=\epsilon} \left[-\frac{u(\mathbf{x})}{4\pi\epsilon^2} - u(\mathbf{x})\frac{\partial H}{\partial s} - \frac{1}{4\pi\epsilon}\frac{\partial u}{\partial s} + H(\mathbf{x})\frac{\partial u}{\partial s}\right] dS \\
&= -\iint_{s=\epsilon} \frac{u(\mathbf{x})}{4\pi\epsilon^2}\, dS - \iint_{s=\epsilon} u(\mathbf{x})\frac{\partial H}{\partial s}\, dS - \iint_{s=\epsilon} \frac{1}{4\pi\epsilon}\frac{\partial u}{\partial s}\, dS + \iint_{s=\epsilon} H(\mathbf{x})\frac{\partial u}{\partial s} dS \\
&= -\int_0^\pi \int_0^{2\pi} \frac{u(\mathbf{x})}{4\pi\epsilon^2}\,(\epsilon^2 \sin\phi\, d\theta\, d\phi) - \int_0^\pi \int_0^{2\pi} u(\mathbf{x})\frac{\partial H}{\partial s}\,(\epsilon^2 \sin\phi\, d\theta\, d\phi) \\
&\qquad - \int_0^\pi \int_0^{2\pi} \frac{1}{4\pi\epsilon}\frac{\partial u}{\partial s}\,(\epsilon^2 \sin\phi\, d\theta\, d\phi) + \int_0^\pi \int_0^{2\pi} H(\mathbf{x})\frac{\partial u}{\partial s}\,(\epsilon^2 \sin\phi\, d\theta\, d\phi) \\
&= -\frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} u(\mathbf{x}) \sin\phi\, d\theta\, d\phi - \epsilon^2 \int_0^\pi \int_0^{2\pi} u(\mathbf{x})\frac{\partial H}{\partial s} \sin\phi\, d\theta\, d\phi \\
&\qquad - \frac{\epsilon}{4\pi}\int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial s} \sin\phi\, d\theta\, d\phi + \epsilon^2 \int_0^\pi \int_0^{2\pi} H(\mathbf{x})\frac{\partial u}{\partial s} \sin\phi\, d\theta\, d\phi
\end{aligned}$$

Take the limit of both sides as $\epsilon \to 0$. As a result, all terms vanish except the first.

$$\begin{aligned}
\lim_{\epsilon\to 0} B_\epsilon &= \lim_{\epsilon\to 0}\left[-\frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} u(\mathbf{x}) \sin\phi\, d\theta\, d\phi - \epsilon^2 \int_0^\pi \int_0^{2\pi} u(\mathbf{x})\frac{\partial H}{\partial s} \sin\phi\, d\theta\, d\phi\right. \\
&\qquad \left. - \frac{\epsilon}{4\pi}\int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial s} \sin\phi\, d\theta\, d\phi + \epsilon^2 \int_0^\pi \int_0^{2\pi} H(\mathbf{x})\frac{\partial u}{\partial s} \sin\phi\, d\theta\, d\phi\right] \\
&= \lim_{\epsilon\to 0}\left[-\frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} u(\mathbf{x}) \sin\phi\, d\theta\, d\phi\right]
\end{aligned}$$

The value of $u(\mathbf{x})$ tends to $u(\mathbf{b})$, a constant, as the radius of the sphere $|\mathbf{x} - \mathbf{b}| = \epsilon$ tends to zero.

$$\begin{aligned}
\lim_{\epsilon\to 0} B_\epsilon &= -\frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} u(\mathbf{b}) \sin\phi\, d\theta\, d\phi \\
&= -\frac{u(\mathbf{b})}{4\pi}\left(\int_0^\pi \sin\phi\, d\phi\right)\left(\int_0^{2\pi} d\theta\right) \\
&= -\frac{u(\mathbf{b})}{4\pi}(2)(2\pi) \\
&= -u(\mathbf{b}) \\
&= -G(\mathbf{b}; \mathbf{a})
\end{aligned}$$