Exercise 1

Find the one-dimensional Green's function for the interval $(0, l)$. The three properties defining it can be restated as follows.

(i) It solves $G''(x) = 0$ for $x \neq x_0$ ("harmonic").

(ii) $G(0) = G(l) = 0$.

(iii) $G(x)$ is continuous at x_0 and $G(x) + \frac{1}{2}|x - x_0|$ is harmonic at x_0.

Solution

The Green's function $G = G(\mathbf{x}; \mathbf{x}_0)$ for the operator $-\Delta$ satisfies the following boundary value problem.

$$\begin{aligned} -\Delta G &= \delta(\mathbf{x} - \mathbf{x}_0) && \text{in } D \\ G &= 0 && \text{on bdy } D \end{aligned}$$

In one dimension on the interval $(0, l)$ this problem becomes

$$\begin{aligned} -\frac{d^2G}{dx^2} &= \delta(x - x_0) && \text{in } (0, l) \\ G &= 0 && \text{at } x = 0 \text{ and } x = l. \end{aligned}$$

If $x \neq x_0$, then $-d^2G/dx^2 = 0$, and the general solution is obtained by integrating both sides with respect to x twice. To satisfy the boundary conditions, there is one solution to the left of $x = x_0$ and one to the right of it.

$$G(x; x_0) = \begin{cases} C_1x + C_2 & \text{if } x < x_0 \\ C_3x + C_4 & \text{if } x > x_0 \end{cases}$$

Four boundary conditions are necessary to determine C_1, C_2, C_3, and C_4. The solution to the left of $x = x_0$ must satisfy $G = 0$ at $x = 0$, and the solution to the right of $x = x_0$ must satisfy $G = 0$ at $x = l$.

$$\begin{aligned} C_1(0) + C_2 &= 0 & \qquad C_3(l) + C_4 &= 0 \\ C_2 &= 0 & C_4 &= -C_3l \end{aligned}$$

For the third condition, we require that the Green's function be continuous at $x = x_0$.

$$\begin{aligned} C_1x_0 + C_2 &= C_3x_0 + C_4 \\ C_1x_0 &= C_3x_0 - C_3l \\ C_1x_0 &= C_3(x_0 - l) \end{aligned} \tag{1}$$

The fourth condition is obtained by integrating both sides of the differential equation over the spike located at $x = x_0$.

$$\begin{aligned} \int_{x_0-\epsilon}^{x_0+\epsilon} -\frac{d^2G}{dx^2}\,dx &= \int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x - x_0)\,dx \\ -\frac{dG}{dx}\bigg|_{x_0+\epsilon} + \frac{dG}{dx}\bigg|_{x_0-\epsilon} &= 1 \\ -(C_3) + (C_1) = 1 \quad &\rightarrow \quad C_1 = 1 + C_3 \end{aligned}$$

Substitute this result for C_1 into equation (1) and solve for C_3.

$$(1 + C_3)x_0 = C_3(x_0 - l) \quad \rightarrow \quad C_3 = -\frac{x_0}{l}$$

So then

$$C_1 = 1 - \frac{x_0}{l} = \frac{l - x_0}{l} \quad \text{and} \quad C_4 = -C_3 l = x_0.$$

Plug in the constants to the formula for the Green's function.

$$G(x; x_0) = \begin{cases} \dfrac{l - x_0}{l}x & \text{if } x < x_0 \\ -\dfrac{x_0}{l}x + x_0 & \text{if } x > x_0 \end{cases}$$

Therefore,

$$G(x; x_0) = \begin{cases} \dfrac{l - x_0}{l}x & \text{if } 0 \leq x < x_0 \\ \dfrac{l - x}{l}x_0 & \text{if } x_0 < x \leq l \end{cases}.$$