Exercise 2

Verify directly from (3) or (4) that the solution of the half-space problem satisfies the condition at infinity:

$$u(\mathbf{x}) \to 0 \qquad \text{as } |\mathbf{x}| \to \infty.$$

Assume that $h(x, y)$ is a continuous function that vanishes outside some circle.

Solution

The solution to the half-space problem,

$$\begin{aligned} \Delta u &= 0 \quad \text{for } z > 0 \\ u(x, y, 0) &= h(x, y), \end{aligned}$$

is given in equation (3) of the text.

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{h(x, y)}{[(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{3/2}} \, dx \, dy$$

Switch the roles of (x_0, y_0, z_0) and (x, y, z): Now (x, y, z) is the point we're interested in evaluating u at, and (x_0, y_0, z_0) is a point in the domain being integrated over in space.

$$u(x, y, z) = \frac{z}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{h(x_0, y_0)}{[(x_0 - x)^2 + (y_0 - y)^2 + z^2]^{3/2}} \, dx_0 \, dy_0$$

In order to verify that u tends to zero as $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$ tends to infinity, a particular example for h will be chosen, the integral will be evaluated, and the limit will be taken. Let $h(x, y)$ be 1 inside a unit square and 0 outside of it.

$$h(x, y) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The double integral, then, is just over this unit square rather than the entire x_0y_0-plane.

$$\begin{aligned} u(x, y, z) &= \frac{z}{2\pi} \int_0^1 \int_0^1 \frac{1}{[(x_0 - x)^2 + (y_0 - y)^2 + z^2]^{3/2}} \, dx_0 \, dy_0 \\ &= \frac{z}{2\pi} \int_0^1 \left\{ \int_0^1 \frac{1}{[(x_0 - x)^2 + (y_0 - y)^2 + z^2]^{3/2}} \, dx_0 \right\} dy_0 \\ &= \frac{z}{2\pi} \int_0^1 \left\{ \frac{x_0 - x}{[(y_0 - y)^2 + z^2]\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + z^2}} \right\} \Bigg|_0^1 dy_0 \\ &= \frac{z}{2\pi} \int_0^1 \frac{1}{(y_0 - y)^2 + z^2} \left\{ \frac{1 - x}{\sqrt{(1 - x)^2 + (y_0 - y)^2 + z^2}} + \frac{x}{\sqrt{x^2 + (y_0 - y)^2 + z^2}} \right\} dy_0 \\ &= \frac{z}{2\pi} \left\{ \frac{1}{z} \left[\tan^{-1} \frac{x(y_0 - y)}{z\sqrt{x^2 + (y_0 - y)^2 + z^2}} - \tan^{-1} \frac{(x - 1)(y_0 - y)}{z\sqrt{(1 - x)^2 + (y_0 - y)^2 + z^2}} \right] \right\} \Bigg|_0^1 \end{aligned}$$

As a result,

$$
\begin{aligned}
u(x,y,z) &= \frac{1}{2\pi}\left[\tan^{-1}\frac{x(1-y)}{z\sqrt{x^2+(1-y)^2+z^2}} - \tan^{-1}\frac{x(-y)}{z\sqrt{x^2+y^2+z^2}}\right. \\
&\qquad \left. - \tan^{-1}\frac{(x-1)(1-y)}{z\sqrt{(1-x)^2+(1-y)^2+z^2}} + \tan^{-1}\frac{(x-1)(-y)}{z\sqrt{(1-x)^2+y^2+z^2}}\right] \\
&= \frac{1}{2\pi}\left[-\tan^{-1}\frac{x(y-1)}{z\sqrt{x^2+(y-1)^2+z^2}} + \tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}\right. \\
&\qquad \left. + \tan^{-1}\frac{(x-1)(y-1)}{z\sqrt{(x-1)^2+(y-1)^2+z^2}} - \tan^{-1}\frac{(x-1)y}{z\sqrt{(x-1)^2+y^2+z^2}}\right].
\end{aligned}
$$

Now take the limit of u as $|\mathbf{x}| \to \infty$.

$$
\begin{aligned}
\lim_{|\mathbf{x}|\to\infty} u(x,y,z) = \lim_{|\mathbf{x}|\to\infty} \frac{1}{2\pi}&\left[-\tan^{-1}\frac{x(y-1)}{z\sqrt{x^2+(y-1)^2+z^2}} + \tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}\right. \\
&\left. + \tan^{-1}\frac{(x-1)(y-1)}{z\sqrt{(x-1)^2+(y-1)^2+z^2}} - \tan^{-1}\frac{(x-1)y}{z\sqrt{(x-1)^2+y^2+z^2}}\right].
\end{aligned}
$$

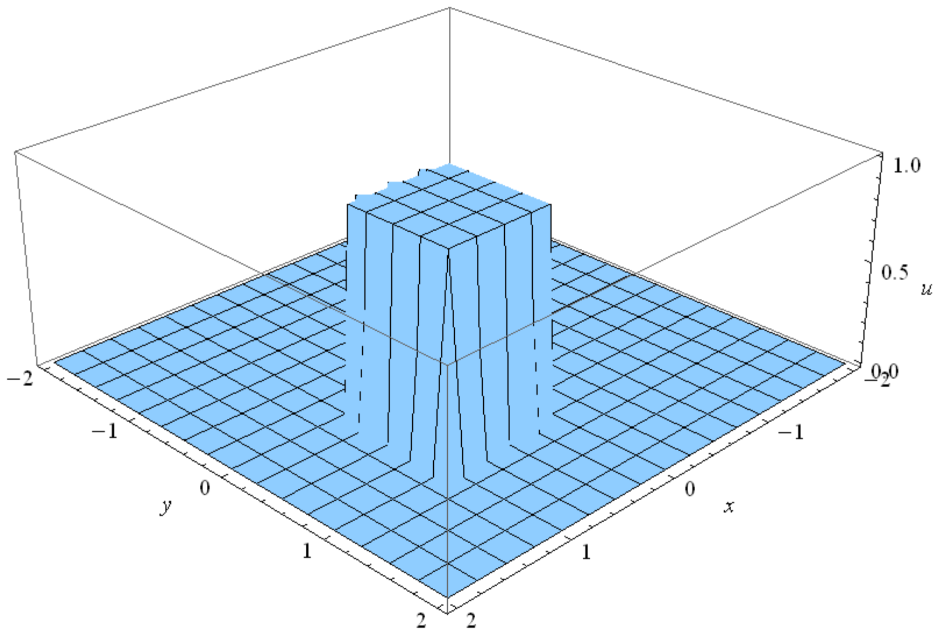
x, y, and z become very large, so the 1s become negligible and all the terms cancel.

$$
\begin{aligned}
\lim_{|\mathbf{x}|\to\infty} u(x,y,z) &= \lim_{|\mathbf{x}|\to\infty} \frac{1}{2\pi}\left(-\tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}} + \tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}\right. \\
&\qquad \left. + \tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}} - \tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}\right) \\
&= \lim_{|\mathbf{x}|\to\infty} \frac{1}{2\pi}(0) \\
&= 0
\end{aligned}
$$

This verifies the condition at infinity. Note that taking the limit of u as $z \to 0$ instead (using Mathematica) results in

$$
\begin{aligned}
\lim_{z\to 0} u(x,y,z) = \frac{1}{4}&\left[-\frac{x(y-1)}{\sqrt{x^2(y-1)^2}} + \frac{xy}{\sqrt{x^2y^2}}\right. \\
&\left. + \frac{(x-1)(y-1)}{\sqrt{(x-1)^2(y-1)^2}} - \frac{(x-1)y}{\sqrt{(x-1)^2y^2}}\right].
\end{aligned}
$$

Plotting this result versus x and y yields the unit square assumed initially for h.



This verifies the condition at $z = 0$. Also, u has derivatives of all orders because it's in terms of arctangent, which has derivatives of all orders.

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$
$$\frac{d^2}{dx^2}\tan^{-1}x = -\frac{2x}{(1+x^2)^2}$$
$$\frac{d^3}{dx^3}\tan^{-1}x = \frac{6x^2-1}{(1+x^2)^3}$$
$$\frac{d^4}{dx^4}\tan^{-1}x = -\frac{24x(x^2-1)}{(1+x^2)^4}$$
$$\vdots$$