Exercise 3

Show directly from (3) that the boundary condition is satisfied: $u(x_0, y_0, z_0) \to h(x_0, y_0)$ as $z_0 \to 0$. Assume $h(x, y)$ is continuous and bounded. [*Hint:* Change variables $s^2 = [(x - x_0)^2 + (y - y_0)^2]/z_0^2$ and use the fact that $\int_0^\infty s(s^2 + 1)^{-3/2}\, ds = 1$.]

Solution

The solution to the half-space problem,

$$\begin{aligned} \Delta u &= 0 \quad \text{for } z > 0 \\ u(x, y, 0) &= h(x, y), \end{aligned}$$

is given in equation (3) of the text.

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{h(x, y)}{[(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{3/2}}\, dx\, dy$$

Switch the roles of (x_0, y_0, z_0) and (x, y, z): Now (x, y, z) is the point we're interested in evaluating u at, and (x_0, y_0, z_0) is a point in the integration space.

$$u(x, y, z) = \frac{z}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{h(x_0, y_0)}{[(x_0 - x)^2 + (y_0 - y)^2 + z^2]^{3/2}}\, dx_0\, dy_0$$

Make the change of variables,

$$\begin{aligned} x_0 - x &= r_0 z \cos\theta_0 \\ y_0 - y &= r_0 z \sin\theta_0. \end{aligned}$$

The resulting Jacobian is

$$\frac{\partial(x_0, y_0)}{\partial(r_0, \theta_0)} = \begin{vmatrix} \frac{\partial x_0}{\partial r_0} & \frac{\partial x_0}{\partial \theta_0} \\ \frac{\partial y_0}{\partial r_0} & \frac{\partial y_0}{\partial \theta_0} \end{vmatrix} = \begin{vmatrix} z\cos\theta_0 & -r_0 z \sin\theta_0 \\ z \sin\theta_0 & r_0 z \cos\theta_0 \end{vmatrix} = r_0 z^2 \cos^2\theta_0 + r_0 z^2 \sin^2\theta_0 = r_0 z^2,$$

which means u becomes

$$\begin{aligned} u(x, y, z) &= \frac{z}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{h(x + r_0 z\cos\theta_0, y + r_0 z \sin\theta_0)}{[(r_0 z\cos\theta_0)^2 + (r_0 z \sin\theta_0)^2 + z^2]^{3/2}}\, r_0 z^2\, dr_0\, d\theta_0 \\ &= \frac{z}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{h(x + r_0 z\cos\theta_0, y + r_0 z \sin\theta_0)}{(r_0^2 z^2 \cos^2\theta_0 + r_0^2 z^2 \sin^2\theta_0 + z^2)^{3/2}}\, r_0 z^2\, dr_0\, d\theta_0 \\ &= \frac{z}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{h(x + r_0 z\cos\theta_0, y + r_0 z \sin\theta_0)}{(r_0^2 z^2 + z^2)^{3/2}}\, r_0 z^2\, dr_0\, d\theta_0 \\ &= \frac{z}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{h(x + r_0 z\cos\theta_0, y + r_0 z \sin\theta_0)}{z^3 (r_0^2 + 1)^{3/2}}\, r_0 z^2\, dr_0\, d\theta_0 \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{h(x + r_0 z\cos\theta_0, y + r_0 z \sin\theta_0)}{(r_0^2 + 1)^{3/2}}\, r_0\, dr_0\, d\theta_0. \end{aligned}$$

Now take the limit of both sides as $z \to 0$.

$$\begin{aligned}
\lim_{z\to 0} u(x,y,z) &= \lim_{z\to 0} \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \frac{h(x + r_0 z\cos\theta_0, y + r_0 z\sin\theta_0)}{(r_0^2+1)^{3/2}}\, r_0\, dr_0\, d\theta_0 \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \lim_{z\to 0} \frac{h(x + r_0 z\cos\theta_0, y + r_0 z\sin\theta_0)}{(r_0^2+1)^{3/2}}\, r_0\, dr_0\, d\theta_0 \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \frac{h(x,y)}{(r_0^2+1)^{3/2}}\, r_0\, dr_0\, d\theta_0 \\
&= \frac{h(x,y)}{2\pi}\int_0^{2\pi}\int_0^{\infty} \frac{1}{(r_0^2+1)^{3/2}}\, r_0\, dr_0\, d\theta_0 \\
&= \frac{h(x,y)}{2\pi}\left(\int_0^{2\pi} d\theta_0\right)\int_0^{\infty} \frac{r_0}{(r_0^2+1)^{3/2}}\, dr_0
\end{aligned}$$

Make the substitution,

$$\begin{aligned}
r_0 &= \tan\alpha_0 \\
dr_0 &= \sec^2\alpha_0\, d\alpha_0,
\end{aligned}$$

in the r_0-integral.

$$\begin{aligned}
\lim_{z\to 0} u(x,y,z) &= \frac{h(x,y)}{2\pi}(2\pi)\int_{\tan^{-1}(0)}^{\tan^{-1}(\infty)} \frac{\tan\alpha_0}{(\tan^2\alpha_0+1)^{3/2}}\, \sec^2\alpha_0\, d\alpha_0 \\
&= h(x,y)\int_0^{\pi/2} \frac{\tan\alpha_0}{(\sec^2\alpha_0)^{3/2}}\, \sec^2\alpha_0\, d\alpha_0 \\
&= h(x,y)\int_0^{\pi/2} \frac{\tan\alpha_0}{\sec^3\alpha_0}\, \sec^2\alpha_0\, d\alpha_0 \\
&= h(x,y)\int_0^{\pi/2} \tan\alpha_0\cos\alpha_0\, d\alpha_0 \\
&= h(x,y)\underbrace{\int_0^{\pi/2} \sin\alpha_0\, d\alpha_0}_{=1} \\
&= h(x,y)
\end{aligned}$$