Exercise 4

Verify directly from (3) that the solution has derivatives of all orders in $\{z > 0\}$. Assume that $h(x, y)$ is a continuous function that vanishes outside some circle. (*Hint:* See Section A.3 for differentiation under an integral sign.)

Solution

The solution to the half-space problem,

$$\Delta u = 0 \quad \text{for } z > 0$$
$$u(x, y, 0) = h(x, y),$$

is given in equation (3) of the text.

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{h(x, y)}{[(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{3/2}} \, dx \, dy$$

Switch the roles of (x_0, y_0, z_0) and (x, y, z): Now (x, y, z) is the point we're interested in evaluating u at, and (x_0, y_0, z_0) is a point in the domain being integrated over in space.

$$u(x, y, z) = \frac{z}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{h(x_0, y_0)}{[(x_0 - x)^2 + (y_0 - y)^2 + z^2]^{3/2}} \, dx_0 \, dy_0$$

In order to verify that u has derivatives of all orders, a particular example for h will be chosen, the integral will be evaluated, and derivatives will be taken. Let $h(x, y)$ be 1 inside a unit square and 0 outside of it.

$$h(x, y) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The double integral, then, is just over this unit square rather than the entire $x_0 y_0$-plane.

$$\begin{aligned}
u(x, y, z) &= \frac{z}{2\pi} \int_0^1 \int_0^1 \frac{1}{[(x_0 - x)^2 + (y_0 - y)^2 + z^2]^{3/2}} \, dx_0 \, dy_0 \\
&= \frac{z}{2\pi} \int_0^1 \left\{ \int_0^1 \frac{1}{[(x_0 - x)^2 + (y_0 - y)^2 + z^2]^{3/2}} \, dx_0 \right\} dy_0 \\
&= \frac{z}{2\pi} \int_0^1 \left\{ \frac{x_0 - x}{[(y_0 - y)^2 + z^2]\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + z^2}} \right\} \Bigg|_0^1 dy_0 \\
&= \frac{z}{2\pi} \int_0^1 \frac{1}{(y_0 - y)^2 + z^2} \left\{ \frac{1 - x}{\sqrt{(1 - x)^2 + (y_0 - y)^2 + z^2}} + \frac{x}{\sqrt{x^2 + (y_0 - y)^2 + z^2}} \right\} dy_0 \\
&= \frac{z}{2\pi} \left\{ \frac{1}{z} \left[\tan^{-1} \frac{x(y_0 - y)}{z\sqrt{x^2 + (y_0 - y)^2 + z^2}} - \tan^{-1} \frac{(x - 1)(y_0 - y)}{z\sqrt{(1 - x)^2 + (y_0 - y)^2 + z^2}} \right] \right\} \Bigg|_0^1
\end{aligned}$$

As a result,

$$\begin{aligned}
u(x,y,z) &= \frac{1}{2\pi}\left[\tan^{-1}\frac{x(1-y)}{z\sqrt{x^2+(1-y)^2+z^2}} - \tan^{-1}\frac{x(-y)}{z\sqrt{x^2+y^2+z^2}}\right. \\
&\qquad \left. - \tan^{-1}\frac{(x-1)(1-y)}{z\sqrt{(1-x)^2+(1-y)^2+z^2}} + \tan^{-1}\frac{(x-1)(-y)}{z\sqrt{(1-x)^2+y^2+z^2}}\right] \\
&= \frac{1}{2\pi}\left[-\tan^{-1}\frac{x(y-1)}{z\sqrt{x^2+(y-1)^2+z^2}} + \tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}\right. \\
&\qquad \left. + \tan^{-1}\frac{(x-1)(y-1)}{z\sqrt{(x-1)^2+(y-1)^2+z^2}} - \tan^{-1}\frac{(x-1)y}{z\sqrt{(x-1)^2+y^2+z^2}}\right].
\end{aligned}$$

In order to simplify u, let

$$f(x,y,z) = \frac{xy}{z\sqrt{x^2+y^2+z^2}}$$

so that

$$u(x,y,z) = \frac{1}{2\pi}\left[-\tan^{-1} f(x,y-1,z) + \tan^{-1} f(x,y,z) + \tan^{-1} f(x-1,y-1,z) - \tan^{-1} f(x-1,y,z)\right].$$

The first derivatives of u are

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{2\pi}\left\{-\frac{\frac{\partial f}{\partial x}(x,y-1,z)}{1+[f(x,y-1,z)]^2} + \frac{\frac{\partial f}{\partial x}(x,y,z)}{1+[f(x,y,z)]^2} + \frac{\frac{\partial f}{\partial x}(x-1,y-1,z)}{1+[f(x-1,y-1,z)]^2} - \frac{\frac{\partial f}{\partial x}(x-1,y,z)}{1+[f(x-1,y,z)]^2}\right\} \\
\frac{\partial u}{\partial y} &= \frac{1}{2\pi}\left\{-\frac{\frac{\partial f}{\partial y}(x,y-1,z)}{1+[f(x,y-1,z)]^2} + \frac{\frac{\partial f}{\partial y}(x,y,z)}{1+[f(x,y,z)]^2} + \frac{\frac{\partial f}{\partial y}(x-1,y-1,z)}{1+[f(x-1,y-1,z)]^2} - \frac{\frac{\partial f}{\partial y}(x-1,y,z)}{1+[f(x-1,y,z)]^2}\right\} \\
\frac{\partial u}{\partial z} &= \frac{1}{2\pi}\left\{-\frac{\frac{\partial f}{\partial z}(x,y-1,z)}{1+[f(x,y-1,z)]^2} + \frac{\frac{\partial f}{\partial z}(x,y,z)}{1+[f(x,y,z)]^2} + \frac{\frac{\partial f}{\partial z}(x-1,y-1,z)}{1+[f(x-1,y-1,z)]^2} - \frac{\frac{\partial f}{\partial z}(x-1,y,z)}{1+[f(x-1,y,z)]^2}\right\}.
\end{aligned}$$

Because the first derivatives of u are sums of rational functions involving f and the first derivatives of f, all subsequent derivatives of u exist, since f itself can be differentiated infinitely many times with respect to any variable.