Exercise 7

(a) If $u(x, y) = f(x/y)$ is a harmonic function, solve the ODE satisfied by f.

(b) Show that $\partial u/\partial r \equiv 0$, where $r = \sqrt{x^2 + y^2}$ as usual.

(c) Suppose that $v(x, y)$ is any function in $\{y > 0\}$ such that $\partial v/\partial r \equiv 0$. Show that $v(x, y)$ is a function of the quotient x/y.

(d) Find the boundary values $\lim_{y\to 0} u(x, y) = h(x)$.

(e) Show that your answer to parts (c) and (d) agrees with the general formula from Exercise 6.

Solution

Part (a)

A harmonic function is a function satisfied by the Laplace equation.

$$\Delta u = 0$$

Expand the Laplacian operator in Cartesian coordinates.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Substitute $u = f(x/y)$.

$$\frac{\partial^2}{\partial x^2} f\left(\frac{x}{y}\right) + \frac{\partial^2}{\partial y^2} f\left(\frac{x}{y}\right) = 0$$

Evaluate the derivatives using the chain rule.

$$\frac{\partial}{\partial x}\left[f'\left(\frac{x}{y}\right)\frac{\partial}{\partial x}\left(\frac{x}{y}\right)\right] + \frac{\partial}{\partial y}\left[f'\left(\frac{x}{y}\right)\frac{\partial}{\partial y}\left(\frac{x}{y}\right)\right] = 0$$

$$\frac{\partial}{\partial x}\left[f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)\right] + \frac{\partial}{\partial y}\left[f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)\right] = 0$$

$$f''\left(\frac{x}{y}\right)\frac{\partial}{\partial x}\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) + f''\left(\frac{x}{y}\right)\frac{\partial}{\partial y}\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) + f'\left(\frac{x}{y}\right)\frac{\partial}{\partial y}\left(-\frac{x}{y^2}\right) = 0$$

$$f''\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)^2 + f''\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)^2 + f'\left(\frac{x}{y}\right)\left(\frac{2x}{y^3}\right) = 0$$

$$\frac{1}{y^2}f''\left(\frac{x}{y}\right) + \frac{x^2}{y^4}f''\left(\frac{x}{y}\right) + \frac{2x}{y^3}f'\left(\frac{x}{y}\right) = 0$$

Multiply both sides by y^2.

$$f''\left(\frac{x}{y}\right) + \frac{x^2}{y^2}f''\left(\frac{x}{y}\right) + \frac{2x}{y}f'\left(\frac{x}{y}\right) = 0$$

Let $z = x/y$.

$$f''(z) + z^2 f''(z) + 2z f'(z) = 0$$

The PDE has reduced to an ODE for $f = f(z)$.

$$(1+z^2)f''(z) = -2zf'(z)$$

Solve it for f''/f'.

$$\frac{f''(z)}{f'(z)} = -\frac{2z}{1+z^2}$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dz}\ln|f'(z)| = -\frac{2z}{1+z^2}$$

The absolute value sign is included just because the logarithm argument can't be negative. Integrate both sides with respect to z.

$$\begin{aligned}\ln|f'(z)| &= -\ln(1+z^2) + C_1 \\ &= \ln(1+z^2)^{-1} + C_1\end{aligned}$$

Exponentiate both sides.

$$\begin{aligned}|f'(z)| &= e^{\ln(1+z^2)^{-1}+C_1} \\ &= e^{\ln(1+z^2)^{-1}}e^{C_1} \\ &= (1+z^2)^{-1}e^{C_1} \\ &= \frac{e^{C_1}}{1+z^2}\end{aligned}$$

Remove the absolute value sign on the left by placing $\pm$ on the right side.

$$f'(z) = \frac{\pm e^{C_1}}{1+z^2}$$

Use a new constant C_2 for $\pm e^{C_1}$.

$$f'(z) = \frac{C_2}{1+z^2}$$

Integrate both sides with respect to z once more.

$$f(z) = C_2\tan^{-1} z + C_3$$

Therefore, since $u(x,y) = f(x/y)$,

$$u(x,y) = C_2\tan^{-1}\left(\frac{x}{y}\right) + C_3.$$

Part (b)

Switch to polar coordinates (r, θ) by substituting $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} u(r,\theta) &= C_2 \tan^{-1}\left(\frac{r\cos\theta}{r\sin\theta}\right) + C_3 \\ &= C_2 \tan^{-1}(\cot\theta) + C_3 \end{aligned}$$

Since u is independent of r,

$$\frac{\partial u}{\partial r} = 0.$$

Part (c)

The relationships between Cartesian and polar coordinates are

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \tan\theta &= \frac{y}{x}. \end{aligned}$$

Invert both sides of this second equation.

$$(\tan\theta)^{-1} = \frac{x}{y}$$

This implies that θ is some function of x/y: $\theta = g(x/y)$. Let $v = v(r,\theta)$. If it satisfies

$$\frac{\partial v}{\partial r} = 0,$$

then both sides can be integrated partially with respect to r to get v.

$$\begin{aligned} v(r,\theta) &= F(\theta) \\ &= F(g(x/y)) \\ &= G(x/y) \end{aligned}$$

Therefore, v is an arbitrary function of x/y.

Part (d)

Apply the boundary condition,

$$\lim_{y\to 0} u(x,y) = h(x),$$

to determine C_2 and C_3 in the solution for u.

$$\lim_{y\to 0} u(x,y) = C_2 \lim_{y\to 0} \tan^{-1}\left(\frac{x}{y}\right) + C_3 = h(x)$$

The limit of inverse tangent depends on the sign of x because $\tan^{-1}(\pm\infty) = \pm\pi/2$.

$$\left.\begin{aligned} C_2\left(-\frac{\pi}{2}\right) + C_3 \quad \text{if } x < 0 \\ C_2\left(\frac{\pi}{2}\right) + C_3 \quad \text{if } x > 0 \end{aligned}\right\} = h(x)$$

Part (e)

In Exercise 6, the solution to

$$\begin{aligned} \Delta u &= f(x,y), \qquad (-\infty < x < \infty,\ y > 0) \\ u(x,0) &= h(x) \end{aligned}$$

was found to be

$$u(x,y) = \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty f(x_0,y_0)\ln\frac{(x_0-x)^2+(y_0-y)^2}{(x_0-x)^2+(y_0+y)^2}\,dx_0\,dy_0 + \frac{y}{\pi}\int_{-\infty}^\infty \frac{h(x_0)}{(x_0-x)^2+y^2}\,dx_0.$$

Make the trigonometric substitution,

$$\begin{aligned} x_0 - x &= y\tan\phi_0 \quad \Rightarrow \quad (x_0-x)^2+y^2 = y^2\sec^2\phi_0 \\ dx_0 &= y\sec^2\phi_0\,d\phi_0, \end{aligned}$$

in the single integral.

$$\begin{aligned} u(x,y) &= \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty f(x_0,y_0)\ln\frac{(x_0-x)^2+(y_0-y)^2}{(x_0-x)^2+(y_0+y)^2}\,dx_0\,dy_0 + \frac{y}{\pi}\int_{\tan^{-1}(-\infty)}^{\tan^{-1}(\infty)} \frac{h(x+y\tan\phi_0)}{y^2\sec^2\phi_0}\,y\sec^2\phi_0\,d\phi_0 \\ &= \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty f(x_0,y_0)\ln\frac{(x_0-x)^2+(y_0-y)^2}{(x_0-x)^2+(y_0+y)^2}\,dx_0\,dy_0 + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} h(x+y\tan\phi_0)\,d\phi_0 \end{aligned}$$

Now take the limit of both sides as $y \to 0$.

$$\begin{aligned} \lim_{y\to 0} u(x,y) &= \lim_{y\to 0}\frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty f(x_0,y_0)\ln\frac{(x_0-x)^2+(y_0-y)^2}{(x_0-x)^2+(y_0+y)^2}\,dx_0\,dy_0 + \lim_{y\to 0}\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} h(x+y\tan\phi_0)\,d\phi_0 \\ &= \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty f(x_0,y_0)\ln\frac{(x_0-x)^2+(y_0)^2}{(x_0-x)^2+(y_0)^2}\,dx_0\,dy_0 + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} h(x)\,d\phi_0 \\ &= \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty f(x_0,y_0)(\ln 1)\,dx_0\,dy_0 + \frac{h(x)}{\pi}\int_{-\pi/2}^{\pi/2} d\phi_0 \\ &= \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty f(x_0,y_0)(0)\,dx_0\,dy_0 + \frac{h(x)}{\pi}(\pi) \\ &= h(x) \end{aligned}$$

The boundary condition is satisfied as $y \to 0$.

For the Laplace equation in the upper half-plane,

$$\begin{aligned} \Delta u &= 0, \qquad (-\infty < x < \infty,\ y > 0) \\ u(x,0) &= h(x), \end{aligned}$$

the solution reduces to

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{h(x_0)}{(x_0 - x)^2 + y^2} \, dx_0.$$

If

$$h(x) = \begin{cases} C_2\left(-\dfrac{\pi}{2}\right) + C_3 & \text{if } x < 0 \\ C_2\left(\dfrac{\pi}{2}\right) + C_3 & \text{if } x > 0 \end{cases},$$

then

$$\begin{aligned} u(x,y) &= \frac{y}{\pi}\left[\int_{-\infty}^{0} \frac{C_2\left(-\frac{\pi}{2}\right) + C_3}{(x_0 - x)^2 + y^2} \, dx_0 + \int_0^{\infty} \frac{C_2\left(\frac{\pi}{2}\right) + C_3}{(x_0 - x)^2 + y^2} \, dx_0\right] \\ &= \frac{y}{\pi}\left[\left(-\frac{\pi}{2}C_2 + C_3\right) \int_{-\infty}^{0} \frac{dx_0}{(x_0 - x)^2 + y^2} + \left(\frac{\pi}{2}C_2 + C_3\right) \int_0^{\infty} \frac{dx_0}{(x_0 - x)^2 + y^2}\right]. \end{aligned}$$

Make the trigonometric substitution,

$$\begin{aligned} x_0 - x &= y \tan\phi_0 \quad \Rightarrow \quad (x_0 - x)^2 + y^2 = y^2 \sec^2\phi_0 \\ dx_0 &= y \sec^2\phi_0 \, d\phi_0. \end{aligned}$$

Consequently,

$$\begin{aligned} u(x,y) &= \frac{y}{\pi}\left[\left(-\frac{\pi}{2}C_2 + C_3\right) \int_{\tan^{-1}(-\infty)}^{\tan^{-1}(-x/y)} \frac{y\sec^2\phi_0 \, d\phi_0}{y^2 \sec^2\phi_0} + \left(\frac{\pi}{2}C_2 + C_3\right) \int_{\tan^{-1}(-x/y)}^{\tan^{-1}(\infty)} \frac{y\sec^2\phi_0 \, d\phi_0}{y^2\sec^2\phi_0}\right] \\ &= \frac{1}{\pi}\left[\left(-\frac{\pi}{2}C_2 + C_3\right) \int_{\tan^{-1}(-\infty)}^{\tan^{-1}(-x/y)} d\phi_0 + \left(\frac{\pi}{2}C_2 + C_3\right) \int_{\tan^{-1}(-x/y)}^{\tan^{-1}(\infty)} d\phi_0\right] \\ &= \frac{1}{\pi}\left\{\left(-\frac{\pi}{2}C_2 + C_3\right)\left[\tan^{-1}\left(-\frac{x}{y}\right) - \tan^{-1}(-\infty)\right] + \left(\frac{\pi}{2}C_2 + C_3\right)\left[\tan^{-1}(\infty) - \tan^{-1}\left(-\frac{x}{y}\right)\right]\right\} \\ &= \frac{1}{\pi}\left\{\left(-\frac{\pi}{2}C_2 + C_3\right)\left[-\tan^{-1}\left(\frac{x}{y}\right) + \frac{\pi}{2}\right] + \left(\frac{\pi}{2}C_2 + C_3\right)\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{x}{y}\right)\right]\right\} \\ &= \frac{1}{\pi}\left[\pi C_2 \tan^{-1}\left(\frac{x}{y}\right) + \pi C_3\right] \\ &= C_2 \tan^{-1}\left(\frac{x}{y}\right) + C_3. \end{aligned}$$

The general formula from Exercise 6 agrees with the result from part (d).