Exercise 9

Find the Green's function for the tilted half-space $\{(x, y, z) : \ ax + by + cz > 0\}$. (*Hint:* Either do it from scratch by reflecting across the tilted plane, or change variables in the double integral (3) using a linear transformation.)

Solution

The aim is to solve Poisson's equation in the tilted half-space above the plane $ax + by + cz = 0$ that is subject to a boundary condition.

$$\begin{aligned} \Delta u &= f(x, y, z), \qquad ax + by + cz > 0 \\ u(x, y, z) &= h(x, y, z) \quad \text{on } ax + by + cz = 0 \end{aligned}$$

A Green's function representation for the solution can be obtained from Green's second identity,

$$\iiint\limits_D (u\Delta v - v\Delta u)\, dV = \iint\limits_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS,$$

which holds for any two functions, u and v, over any domain and its boundary. Let v be the Green's function: $v = G$.

$$\iiint\limits_D (u\Delta G - G\Delta u)\, dV = \iint\limits_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \right) dS \tag{1}$$

If we require $G = G(x, y, z; x_0, y_0, z_0)$ to satisfy

$$\begin{aligned} \Delta G &= \delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \qquad ax + by + cz > 0 \\ G &= 0 \quad \text{on } ax + by + cz = 0, \end{aligned}$$

where (x_0, y_0, z_0) is a point in the tilted half-space, then equation (1) becomes

$$\iiint\limits_D [u(x, y, z)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) - G(x, y, z; x_0, y_0, z_0) f(x, y, z)]\, dV = \iint\limits_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - (0)\frac{\partial u}{\partial n} \right) dS.$$

Write the normal derivative $\partial G/\partial n$ as $\nabla G \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the outward unit vector normal to the plane.

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\frac{1}{c}(ax+by)}^{\infty} [u(x, y, z)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) - G(x, y, z; x_0, y_0, z_0) f(x, y, z)]\, dz\, dy\, dx \\ &\qquad = \iint\limits_{\text{bdy } D} h(x, y, z)\nabla G \cdot \hat{\mathbf{n}}\, dS \\ &\qquad = \iint\limits_{\text{bdy } D} h(x, y, z(x, y))\nabla G \cdot \frac{\langle a, b, c\rangle}{\sqrt{a^2 + b^2 + c^2}} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \\ &\qquad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y, z(x, y))\nabla G \cdot \frac{\langle a, b, c\rangle}{\sqrt{a^2 + b^2 + c^2}} \sqrt{\left(-\frac{a}{c}\right)^2 + \left(-\frac{b}{c}\right)^2 + 1}\, dy\, dx \end{aligned}$$

Since (x_0, y_0, z_0) is a point inside the tilted half-space, the integral on the left involving u becomes $u(x_0, y_0, z_0)$. Simplify the right side.

$$\begin{aligned} u(x_0,y_0,z_0) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\frac{1}{c}(ax+by)}^{\infty} G(x,y,z;x_0,y_0,z_0)f(x,y,z)\,dz\,dy\,dx \\ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y,z(x,y))\nabla G\cdot\frac{\langle a,b,c\rangle}{\sqrt{a^2+b^2+c^2}}\sqrt{\frac{a^2+b^2+c^2}{c^2}}\,dy\,dx \\ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y,z(x,y))\left\langle\frac{\partial G}{\partial x},\frac{\partial G}{\partial y},\frac{\partial G}{\partial z}\right\rangle\cdot\langle a,b,c\rangle\left(\frac{1}{c}\right)dy\,dx \\ = \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[h(x,y,z)\left(a\frac{\partial G}{\partial x}+b\frac{\partial G}{\partial y}+c\frac{\partial G}{\partial z}\right)\right]\Bigg|_{z=-\frac{1}{c}(ax+by)}dy\,dx \end{aligned}$$

Solve this equation for u.

$$\begin{aligned} u(x_0,y_0,z_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\frac{1}{c}(ax+by)}^{\infty} G(x,y,z;x_0,y_0,z_0)f(x,y,z)\,dz\,dy\,dx \\ + \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[h(x,y,z)\left(a\frac{\partial G}{\partial x}+b\frac{\partial G}{\partial y}+c\frac{\partial G}{\partial z}\right)\right]\Bigg|_{z=-\frac{1}{c}(ax+by)}dy\,dx \end{aligned}$$

Switch the roles of x_0, y_0, and z_0 with those of x, y, and z, respectively.

$$\begin{aligned} u(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\frac{1}{c}(ax_0+by_0)}^{\infty} G(x_0,y_0,z_0;x,y,z)f(x_0,y_0,z_0)\,dz_0\,dy_0\,dx_0 \\ + \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[h(x_0,y_0,z_0)\left(a\frac{\partial G}{\partial x_0}+b\frac{\partial G}{\partial y_0}+c\frac{\partial G}{\partial z_0}\right)\right]\Bigg|_{z_0=-\frac{1}{c}(ax_0+by_0)}dy_0\,dx_0 \end{aligned}$$

Therefore, using the fact that the Green's function is symmetric,

$$\begin{aligned} u(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\frac{1}{c}(ax_0+by_0)}^{\infty} G(x,y,z;x_0,y_0,z_0)f(x_0,y_0,z_0)\,dz_0\,dy_0\,dx_0 \\ + \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[h(x_0,y_0,z_0)\left(a\frac{\partial G}{\partial x_0}+b\frac{\partial G}{\partial y_0}+c\frac{\partial G}{\partial z_0}\right)\right]\Bigg|_{z_0=-\frac{1}{c}(ax_0+by_0)}dy_0\,dx_0. \end{aligned}$$

The solution for Poisson's equation is known, then, if the Green's function in the tilted half-space can be determined. Begin by finding the Green's function in infinite space (no boundaries).

$$\Delta g = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \qquad -\infty < x,y,z < \infty$$

g can be interpreted as the electrostatic potential, and $\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$ can be interpreted as the charge density for a unit charge located at (x_0, y_0, z_0). Since there are no boundaries, g is expected to vary solely as a function of the radial distance from (x_0, y_0, z_0): $g = g(r)$, where $r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$. Integrate both sides over a solid ball centered at (x_0, y_0, z_0) with radius r.

$$\iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le r^2}} \Delta g\,dV = \iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le r^2}} \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\,dV$$

Since the ball contains (x_0, y_0, z_0), the right side is 1. Write the Laplacian operator Δ as ∇^2

$$\iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\leq \mathscr{r}^2}} \nabla^2 g\, dV = 1$$

$$\iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\leq \mathscr{r}^2}} \nabla\cdot\nabla g\, dV = 1$$

and apply the divergence theorem.

$$\iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2= \mathscr{r}^2}} \nabla g\cdot\hat{\mathscr{r}}\, dS = 1$$

Here $\hat{\mathscr{r}}$ is the unit vector normal to this ball at every point on the boundary.

$$\iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2= \mathscr{r}^2}} \frac{dg}{d\mathscr{r}}\, dS = 1$$

Because g only depends on $\mathscr{r}$, its derivative is constant on the ball's boundary.

$$\frac{dg}{d\mathscr{r}}\iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2= \mathscr{r}^2}} dS = 1$$

This surface integral is just the ball's surface area.

$$\frac{dg}{d\mathscr{r}}(4\pi\mathscr{r}^2) = 1$$

Divide both sides by $4\pi\mathscr{r}^2$.

$$\frac{dg}{d\mathscr{r}} = \frac{1}{4\pi\mathscr{r}^2}$$

Integrate both sides with respect to $\mathscr{r}$.

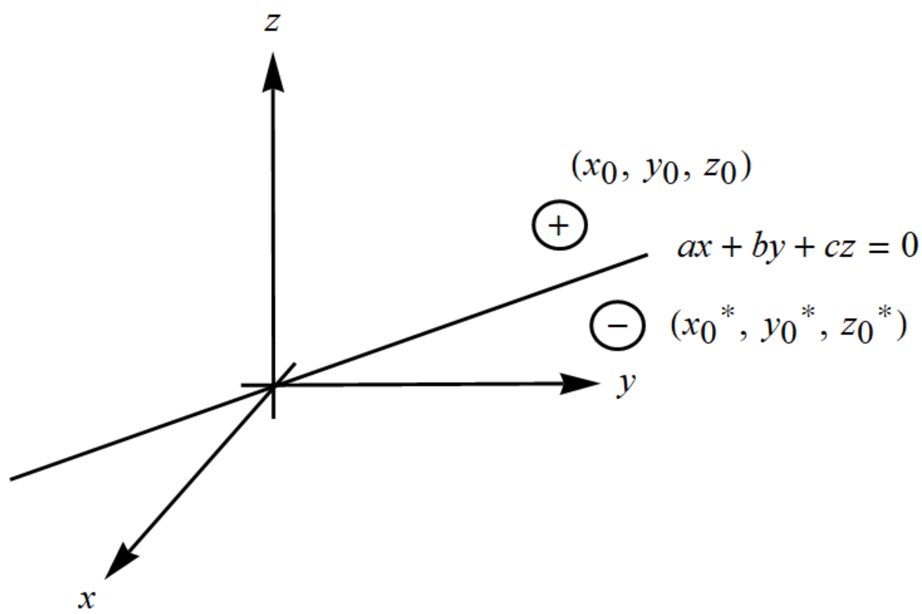
$$g(\mathscr{r}) = -\frac{1}{4\pi\mathscr{r}}$$

The infinite-space Green's function is then

$$g(x,y,z;x_0,y_0,z_0) = -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}.$$

Now that it's known, the Green's function for the tilted half-space can be determined by the method of images. A convocation of point charges in space will be arranged so that the boundary condition, $G = 0$ on $ax + by + cz = 0$, is satisfied.

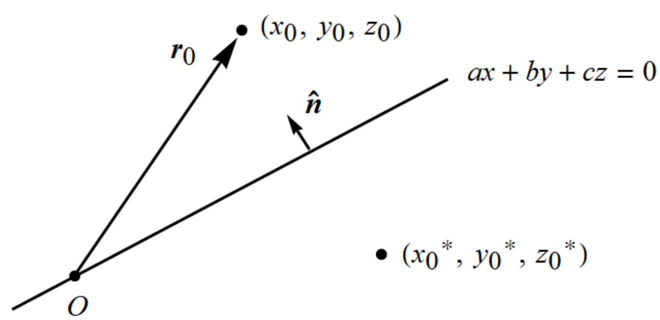
If a positive unit charge is located at (x_0, y_0, z_0), then the plane $ax + by + cz = 0$ can be made to have zero potential at every point by placing a negative unit charge at (x_0^*, y_0^*, z_0^*), a point that lies as far from the plane as (x_0, y_0, z_0).



The tilted half-space Green's function can now be written.

$$G(x, y, z; x_0, y_0, z_0) = +g(x, y, z; x_0, y_0, z_0) - g(x, y, z; x_0^*, y_0^*, z_0^*), \quad ax + by + cz > 0$$

Since g is defined over all of space, it's important to note the restriction to $ax + by + cz > 0$ for G. The goal now is to determine x_0^*, y_0^*, and z_0^* in terms of x_0, y_0, and z_0.

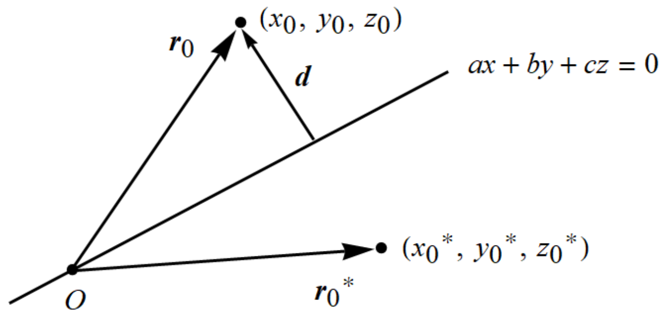


The perpendicular distance d from the plane to (x_0, y_0, z_0) is given by the dot product of the position vector $\mathbf{r}_0$ with $\hat{\mathbf{n}}$.

$$\begin{aligned} d &= \mathbf{r}_0 \cdot \hat{\mathbf{n}} \\ &= \langle x_0, y_0, z_0 \rangle \cdot \frac{\langle a, b, c \rangle}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{ax_0 + by_0 + cz_0}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

Multiply this quantity by $\hat{\mathbf{n}}$ to turn it into a position vector from the plane to (x_0, y_0, z_0).

$$\begin{aligned}\mathbf{d} &= (\mathbf{r}_0 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \\ &= \frac{ax_0 + by_0 + cz_0}{\sqrt{a^2+b^2+c^2}} \frac{\langle a, b, c\rangle}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{ax_0 + by_0 + cz_0}{a^2+b^2+c^2}\langle a, b, c\rangle\end{aligned}$$



The position vector from the origin to (x_0^*, y_0^*, z_0^*) is then

$$\begin{aligned}\mathbf{r}_0^* &= \mathbf{r}_0 - 2\mathbf{d} \\ &= \langle x_0, y_0, z_0\rangle - 2\frac{ax_0 + by_0 + cz_0}{a^2+b^2+c^2}\langle a, b, c\rangle \\ &= \left\langle x_0 - 2a\frac{ax_0 + by_0 + cz_0}{a^2+b^2+c^2}, y_0 - 2b\frac{ax_0 + by_0 + cz_0}{a^2+b^2+c^2}, z_0 - 2c\frac{ax_0 + by_0 + cz_0}{a^2+b^2+c^2}\right\rangle \\ &= \langle x_0^*, y_0^*, z_0^*\rangle.\end{aligned}$$

Therefore, the tilted half-space Green's function is

$$\begin{aligned}
G(x,y,z;x_0,y_0,z_0) &= +g(x,y,z;x_0,y_0,z_0) - g(x,y,z;x_0^*,y_0^*,z_0^*) \\
&= g(x,y,z;x_0,y_0,z_0) - g\left(x,y,z;x_0 - 2a\frac{ax_0+by_0+cz_0}{a^2+b^2+c^2}, y_0 - 2b\frac{ax_0+by_0+cz_0}{a^2+b^2+c^2}, z_0 - 2c\frac{ax_0+by_0+cz_0}{a^2+b^2+c^2}\right) \\
&= -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} \\
&\qquad + \frac{1}{4\pi\sqrt{\left[x-\left(x_0-2a\frac{ax_0+by_0+cz_0}{a^2+b^2+c^2}\right)\right]^2+\left[y-\left(y_0-2b\frac{ax_0+by_0+cz_0}{a^2+b^2+c^2}\right)\right]^2+\left[z-\left(z_0-2c\frac{ax_0+by_0+cz_0}{a^2+b^2+c^2}\right)\right]^2}}.
\end{aligned}$$