

Exercise 12

Find the potential of the electrostatic field due to a point charge located outside a grounded sphere. (*Hint:* This is just the Green's function for the exterior of the sphere. Find it by the method of reflection.)

Solution

The aim here is to solve the Poisson equation outside a solid ball with radius R that is subject to a boundary condition. Use a spherical coordinate system (ρ, ϕ, θ) in which θ is the angle from the polar axis.

$$\begin{aligned}\Delta u &= f(\rho, \phi, \theta), \quad \rho > R, \quad 0 < \phi < 2\pi, \quad 0 < \theta < \pi \\ u(R, \phi, \theta) &= F(\phi, \theta)\end{aligned}$$

A Green's function representation for the solution can be obtained from Green's second identity,

$$\iiint_D (u\Delta v - v\Delta u) dV = \iint_{\text{bdy } D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS,$$

which holds for any two functions, u and v , over any domain and its boundary. Let v be the Green's function: $v = G = G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0)$.

$$\iiint_D (u\Delta G - G\Delta u) dV = \iint_{\text{bdy } D} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) dS \quad (1)$$

If we require it to satisfy

$$\begin{aligned}\Delta G &= \delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(\theta - \theta_0), \quad \rho > R, \quad 0 < \phi < 2\pi, \quad 0 < \theta < \pi \\ G &= 0 \text{ at } \rho = R,\end{aligned}$$

where $(\rho_0, \phi_0, \theta_0)$ is a point outside the ball, then equation (1) becomes

$$\begin{aligned}\iiint_D [u(\rho, \phi, \theta)\delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(\theta - \theta_0) - G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0)f(\rho, \phi, \theta)] dV \\ = \iint_{\text{bdy } D} \left(u \frac{\partial G}{\partial n} - (0) \frac{\partial u}{\partial n} \right) dS.\end{aligned}$$

Since the domain is everywhere exterior to a ball centered at the origin, the outward unit normal vector is $\hat{\mathbf{n}} = -\hat{\boldsymbol{\rho}}$, which means the normal derivative is $\partial/\partial n = -\partial/\partial \rho$.

$$\iiint_D u(\rho, \phi, \theta)\delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(\theta - \theta_0) dV - \iiint_D G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0)f(\rho, \phi, \theta) dV = \iint_{\text{bdy } D} u \left(-\frac{\partial G}{\partial \rho} \right) dS$$

The integral involving the delta functions is $u(\rho_0, \phi_0, \theta_0)$.

$$\begin{aligned}u(\rho_0, \phi_0, \theta_0) - \int_0^\pi \int_0^{2\pi} \int_R^\infty G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0)f(\rho, \phi, \theta)\rho^2 \sin \theta d\rho d\phi d\theta \\ = - \int_0^\pi \int_0^{2\pi} u(R, \phi, \theta) \frac{\partial G}{\partial \rho} \Big|_{\rho=R} R^2 \sin \theta d\phi d\theta\end{aligned}$$

Solve for u .

$$u(\rho_0, \phi_0, \theta_0) = \int_0^\pi \int_0^{2\pi} \int_R^\infty G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho, \phi, \theta) \rho^2 \sin \theta \, d\rho \, d\phi \, d\theta \\ - R^2 \int_0^\pi \int_0^{2\pi} F(\phi, \theta) \frac{\partial G}{\partial \rho} \Big|_{\rho=R} \sin \theta \, d\phi \, d\theta$$

Switch the roles of ρ_0 , ϕ_0 , and θ_0 with those of ρ , ϕ , and θ , respectively.

$$u(\rho, \phi, \theta) = \int_0^\pi \int_0^{2\pi} \int_R^\infty G(\rho_0, \phi_0, \theta_0; \rho, \phi, \theta) f(\rho_0, \phi_0, \theta_0) \rho_0^2 \sin \theta_0 \, d\rho_0 \, d\phi_0 \, d\theta_0 \\ - R^2 \int_0^\pi \int_0^{2\pi} F(\phi_0, \theta_0) \frac{\partial G}{\partial \rho_0} \Big|_{\rho_0=R} \sin \theta_0 \, d\phi_0 \, d\theta_0$$

Therefore, using the fact that the Green's function is symmetric,

$$u(\rho, \phi, \theta) = \int_0^\pi \int_0^{2\pi} \int_R^\infty G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho_0, \phi_0, \theta_0) \rho_0^2 \sin \theta_0 \, d\rho_0 \, d\phi_0 \, d\theta_0 \\ - R^2 \int_0^\pi \int_0^{2\pi} F(\phi_0, \theta_0) \frac{\partial G}{\partial \rho_0} \Big|_{\rho_0=R} \sin \theta_0 \, d\phi_0 \, d\theta_0.$$

The solution for Poisson's equation is known, then, if the Green's function outside of the ball can be determined. Begin by finding the Green's function in infinite space (no boundaries).

$$\Delta g = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \quad -\infty < x, y, z < \infty$$

g can be interpreted as the electrostatic potential, and $\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$ can be interpreted as the charge density for a unit charge located at (x_0, y_0, z_0) . Since there are no boundaries, g is expected to vary solely as a function of the radial distance from (x_0, y_0, z_0) : $g = g(\boldsymbol{z})$, where $\boldsymbol{z} = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$. Integrate both sides over a solid ball centered at (x_0, y_0, z_0) with radius \boldsymbol{z} .

$$\iiint_{\substack{(x-x_0)^2 + (y-y_0)^2 \\ + (z-z_0)^2 \leq \boldsymbol{z}^2}} \Delta g \, dV = \iiint_{\substack{(x-x_0)^2 + (y-y_0)^2 \\ + (z-z_0)^2 \leq \boldsymbol{z}^2}} \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \, dV$$

Since the ball contains (x_0, y_0, z_0) , the right side is 1. Write the Laplacian operator Δ as ∇^2

$$\iiint_{\substack{(x-x_0)^2 + (y-y_0)^2 \\ + (z-z_0)^2 \leq \boldsymbol{z}^2}} \nabla^2 g \, dV = 1$$

$$\iiint_{\substack{(x-x_0)^2 + (y-y_0)^2 \\ + (z-z_0)^2 \leq \boldsymbol{z}^2}} \nabla \cdot \nabla g \, dV = 1$$

and apply the divergence theorem.

$$\iint_{\substack{(x-x_0)^2 + (y-y_0)^2 \\ + (z-z_0)^2 = \boldsymbol{z}^2}} \nabla g \cdot \hat{\boldsymbol{z}} \, dS = 1$$

Here $\hat{\mathbf{z}}$ is the unit vector normal to this ball at every point on the boundary.

$$\iint_{\substack{(x-x_0)^2 + (y-y_0)^2 \\ + (z-z_0)^2 = z^2}} \frac{dg}{dz} dS = 1$$

Because g only depends on z , its derivative is constant on the ball's boundary.

$$\frac{dg}{dz} \iint_{\substack{(x-x_0)^2 + (y-y_0)^2 \\ + (z-z_0)^2 = z^2}} dS = 1$$

This surface integral is just the ball's surface area.

$$\frac{dg}{dz} (4\pi z^2) = 1$$

Divide both sides by $4\pi z^2$.

$$\frac{dg}{dz} = \frac{1}{4\pi z^2}$$

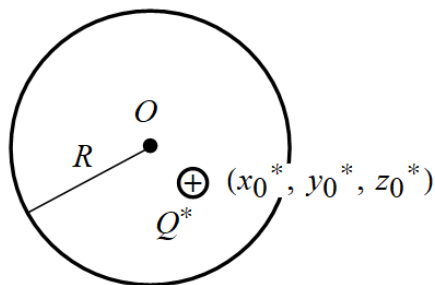
Integrate both sides with respect to z .

$$g(z) = -\frac{1}{4\pi z}$$

The infinite-space Green's function is then

$$g(x, y, z; x_0, y_0, z_0) = -\frac{1}{4\pi \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}.$$

Now that it's known, the Green's function outside the ball can be determined by the method of images. A convocation of point charges in infinite space will be arranged so that the boundary condition, $G = 0$ along $\rho = R$, is satisfied.



$$\oplus (x_0, y_0, z_0)$$

For a positive unit charge located at (x_0, y_0, z_0) outside the ball, place a charge Q^* at (x_0^*, y_0^*, z_0^*) inside the ball such that the charges are collinear with the origin. x_0^* , y_0^* , z_0^* , and Q^* are all unknown at the moment.

Write the Green's function outside the ball (valid for $x^2 + y^2 + z^2 > R^2$).

$$\begin{aligned}
 G(x, y, z; x_0, y_0, z_0) &= +g(x, y, z; x_0, y_0, z_0) + Q^*g(x, y, z; x_0^*, y_0^*, z_0^*) \\
 &= -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} - \frac{Q^*}{4\pi\sqrt{(x-x_0^*)^2+(y-y_0^*)^2+(z-z_0^*)^2}} \\
 &= -\frac{1}{4\pi} \left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \frac{Q^*}{\sqrt{(x-x_0^*)^2+(y-y_0^*)^2+(z-z_0^*)^2}} \right] \\
 &= -\frac{1}{4\pi} \left[\frac{1}{\sqrt{(x^2+y^2+z^2)+(x_0^2+y_0^2+z_0^2)-2(xx_0+yy_0+zz_0)}} \right. \\
 &\quad \left. + \frac{Q^*}{\sqrt{(x^2+y^2+z^2)+(x_0^{*2}+y_0^{*2}+z_0^{*2})-2(xx_0^*+yy_0^*+zz_0^*)}} \right] \quad (2)
 \end{aligned}$$

$$\begin{aligned}
 G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) &= -\frac{1}{4\pi} \left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2(\mathbf{x}\cdot\mathbf{x}_0)}} + \frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2(\mathbf{x}\cdot\mathbf{x}_0^*)}} \right] \\
 &= -\frac{1}{4\pi} \left(\frac{1}{\sqrt{\rho^2+\rho_0^2-2|\mathbf{x}||\mathbf{x}_0|\cos\alpha}} + \frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2|\mathbf{x}||\mathbf{x}_0^*|\cos\alpha}} \right) \\
 &= -\frac{1}{4\pi} \left(\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos\alpha}} + \frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*\cos\alpha}} \right) \\
 &= -\frac{1}{4\pi} \left(\frac{1}{\rho\sqrt{1+\frac{\rho_0^2}{\rho^2}-\frac{2\rho_0}{\rho}\cos\alpha}} + \frac{Q^*}{\rho_0^*\sqrt{1+\frac{\rho_0^{*2}}{\rho_0^2}+1-\frac{2\rho}{\rho_0^*}\cos\alpha}} \right)
 \end{aligned}$$

Here α represents the angle between \mathbf{x} , the position vector of the point we're interested in knowing the potential at, and \mathbf{x}_0 , the position vector of the positive unit charge. Because this positive unit charge and the image charge Q^* are collinear, α is also the angle between \mathbf{x} and \mathbf{x}_0^* , the position vector of the image. The potential at $\rho = R$ is zero.

$$G(R, \phi, \theta; \rho_0, \phi_0, \theta_0) = -\frac{1}{4\pi} \left(\frac{1}{R\sqrt{1+\frac{\rho_0^2}{R^2}-\frac{2\rho_0}{R}\cos\alpha}} + \frac{Q^*}{\rho_0^*\sqrt{1+\frac{R^2}{\rho_0^{*2}}-\frac{2R}{\rho_0^*}\cos\alpha}} \right) = 0$$

In order for the quantity in parentheses to vanish, set

$$\frac{\rho_0}{R} = \frac{R}{\rho_0^*} \quad \text{and} \quad \frac{1}{R} + \frac{Q^*}{\rho_0^*} = 0,$$

which means

$$\begin{aligned}
 \rho_0^* &= \frac{R^2}{\rho_0} \quad \text{and} \quad Q^* = -\frac{\rho_0^*}{R} \\
 & \quad \quad \quad Q^* = -\frac{R}{\rho_0}.
 \end{aligned}$$

Now write equation (2) in terms of spherical coordinates and then substitute these formulas for ρ_0^* and Q^* .

$$\begin{aligned}
 G(x, y, z; x_0, y_0, z_0) &= -\frac{1}{4\pi} \left[\frac{1}{\sqrt{(x^2 + y^2 + z^2) + (x_0^2 + y_0^2 + z_0^2) - 2(xx_0 + yy_0 + zz_0)}} \right. \\
 &\quad \left. + \frac{Q^*}{\sqrt{(x^2 + y^2 + z^2) + (x_0^{*2} + y_0^{*2} + z_0^{*2}) - 2(xx_0^* + yy_0^* + zz_0^*)}} \right] \\
 G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) &= -\frac{1}{4\pi} \left\{ \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2[(\rho \cos \phi \sin \theta)(\rho_0 \cos \phi_0 \sin \theta_0) + (\rho \sin \phi \sin \theta)(\rho_0 \sin \phi_0 \sin \theta_0) + (\rho \cos \theta)(\rho_0 \cos \theta_0)]}} \right. \\
 &\quad \left. + \frac{Q^*}{\sqrt{\rho^2 + \rho_0^{*2} - 2[(\rho \cos \phi \sin \theta)(\rho_0^* \cos \phi_0 \sin \theta_0) + (\rho \sin \phi \sin \theta)(\rho_0^* \sin \phi_0 \sin \theta_0) + (\rho \cos \theta)(\rho_0^* \cos \theta_0)]}} \right\} \\
 &= -\frac{1}{4\pi} \left[\frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0(\cos \phi \cos \phi_0 \sin \theta \sin \theta_0 + \sin \phi \sin \phi_0 \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0)}} \right. \\
 &\quad \left. + \frac{Q^*}{\sqrt{\rho^2 + \rho_0^{*2} - 2\rho\rho_0^*(\cos \phi \cos \phi_0 \sin \theta \sin \theta_0 + \sin \phi \sin \phi_0 \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0)}} \right] \\
 &= -\frac{1}{4\pi} \left[\frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0[\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}} \right. \\
 &\quad \left. + \frac{Q^*}{\sqrt{\rho^2 + \rho_0^{*2} - 2\rho\rho_0^*[\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}} \right] \\
 &= -\frac{1}{4\pi} \left[\frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0[\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}} \right. \\
 &\quad \left. + \frac{-\frac{R}{\rho_0}}{\sqrt{\rho^2 + \frac{R^4}{\rho_0^2} - 2\rho\frac{R^2}{\rho_0}[\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}} \right] \\
 &= -\frac{1}{4\pi} \left[\frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0[\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}} \right. \\
 &\quad \left. - \frac{R}{\sqrt{\rho^2 \rho_0^2 + R^4 - 2R^2 \rho \rho_0[\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}} \right]
 \end{aligned}$$

Therefore, the Green's function outside a ball of radius R in spherical coordinates is

$$G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) = \frac{1}{4\pi} \left[\frac{R}{\sqrt{\rho^2 \rho_0^2 + R^4 - 2R^2 \rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}} - \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}} \right].$$

Calculate $\partial G / \partial \rho_0$

$$\frac{\partial G}{\partial \rho_0} = \frac{1}{4\pi} \left\{ -\frac{R}{2} \frac{2\rho^2 \rho_0 - 2R^2 \rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}{\{\rho^2 \rho_0^2 + R^4 - 2R^2 \rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]\}^{3/2}} + \frac{1}{2} \frac{2\rho_0 - 2\rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}{\{\rho^2 + \rho_0^2 - 2\rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]\}^{3/2}} \right\}$$

and then evaluate it at $\rho_0 = R$.

$$\begin{aligned} \frac{\partial G}{\partial \rho_0} \Big|_{\rho_0=R} &= \frac{1}{4\pi} \left\{ -\frac{R}{2} \frac{2R\rho^2 - 2R^2 \rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}{R^3 \{\rho^2 + R^2 - 2R\rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]\}^{3/2}} + \frac{1}{2} \frac{2R - 2\rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}{\{\rho^2 + R^2 - 2R\rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]\}^{3/2}} \right\} \\ &= \frac{1}{4\pi} \frac{-\frac{\rho^2}{R} + R}{\{\rho^2 + R^2 - 2R\rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]\}^{3/2}} \\ &= \frac{1}{4\pi R} \frac{R^2 - \rho^2}{\{\rho^2 + R^2 - 2R\rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]\}^{3/2}} \end{aligned}$$

Therefore, the solution to the Poisson equation outside a ball with a prescribed boundary condition at $\rho = R$ is

$$u(\rho, \phi, \theta) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \int_R^\infty \left[\frac{R}{\sqrt{\rho^2 \rho_0^2 + R^4 - 2R^2 \rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}} \right. \\ \left. - \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}} \right] f(\rho_0, \phi_0, \theta_0) \rho_0^2 \sin \theta_0 d\rho_0 d\phi_0 d\theta_0 \\ - \frac{R}{4\pi} \int_0^\pi \int_0^{2\pi} F(\phi_0, \theta_0) \frac{R^2 - \rho^2}{\{\rho^2 + R^2 - 2R\rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]\}^{3/2}} \sin \theta_0 d\phi_0 d\theta_0.$$