Exercise 13

Find the Green's function for the half-ball $D = \{x^2 + y^2 + z^2 < a^2,\ z > 0\}$. (*Hint:* The easiest method is to use the solution for the whole ball and reflect it across the plane.)

Solution

The aim here is to solve the Poisson equation inside a half-ball with radius R that is subject to boundary conditions on its spherical and planar surfaces. Use a spherical coordinate system (ρ, ϕ, θ) in which θ is the angle from the polar axis.

$$\begin{aligned} &\Delta u = f(\rho, \phi, \theta), \quad \rho < R,\ 0 < \phi < 2\pi,\ 0 < \theta < \frac{\pi}{2} \\ &u(R, \phi, \theta) = F(\phi, \theta) \\ &u\left(\rho, \phi, \frac{\pi}{2}\right) = H(\rho, \phi) \end{aligned}$$

A Green's function representation for the solution can be obtained from Green's second identity,

$$\iiint_D (u\Delta v - v\Delta u)\, dV = \iint_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS,$$

which holds for any two functions, u and v, over any domain and its boundary. Let v be the Green's function: $v = G = G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0)$.

$$\iiint_D (u\Delta G - G\Delta u)\, dV = \iint_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS \tag{1}$$

If we require it to satisfy

$$\begin{aligned} &\Delta G = \delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(\theta - \theta_0), \qquad \rho < R,\ 0 < \phi < 2\pi,\ 0 < \theta < \frac{\pi}{2} \\ &G = 0 \text{ on bdy } D, \end{aligned}$$

where $(\rho_0, \phi_0, \theta_0)$ is a point in the half-ball, then equation (1) becomes

$$\begin{aligned} \iiint_D [u(\rho, \phi, \theta)\delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(\theta - \theta_0) - G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho, \phi, \theta)]\, dV \\ = \iint_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - (0)\frac{\partial u}{\partial n}\right) dS. \end{aligned}$$

Since the domain is a half-ball centered at the origin, there are two boundaries to consider. There's the spherical boundary $\rho = R$, where the outward unit normal vector is $\hat{\mathbf{n}} = \hat{\boldsymbol{\rho}}$ and the normal derivative is $\partial/\partial n = \partial/\partial \rho$. There's also the planar boundary $z = 0$, where the outward unit normal vector is $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$ and the normal derivative is $\partial/\partial n = -\partial/\partial z$.

$$\iiint_D u(\rho, \phi, \theta)\delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(\theta - \theta_0)\, dV - \iiint_D G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho, \phi, \theta)\, dV = \iint_{\text{bdy } D} u\frac{\partial G}{\partial n}\, dS$$

The integral involving the delta functions is $u(\rho_0, \phi_0, \theta_0)$.

$$\begin{aligned} u(\rho_0, \phi_0, \theta_0) - \int_0^{\pi/2} \int_0^{2\pi} \int_0^R G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho, \phi, \theta) \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta \\ = \int_0^{\pi/2} \int_0^{2\pi} u(R, \phi, \theta) \left. \frac{\partial G}{\partial \rho} \right|_{\rho=R} R^2 \sin\theta \, d\phi \, d\theta + \int_0^{2\pi} \int_0^R u\left(\rho, \phi, \frac{\pi}{2}\right) \left(-\frac{\partial G}{\partial z}\right)\bigg|_{z=0} \rho \, d\rho \, d\phi \end{aligned}$$

Solve for u.

$$\begin{aligned} u(\rho_0, \phi_0, \theta_0) = \int_0^{\pi/2} \int_0^{2\pi} \int_0^R G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho, \phi, \theta) \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta \\ + R^2 \int_0^{\pi/2} \int_0^{2\pi} F(\phi, \theta) \left. \frac{\partial G}{\partial \rho} \right|_{\rho=R} \sin\theta \, d\phi \, d\theta \\ - \int_0^{2\pi} \int_0^R H(\rho, \phi) \left. \frac{\partial G}{\partial z} \right|_{z=0} \rho \, d\rho \, d\phi \end{aligned}$$

In spherical coordinates, $x = \rho \cos\phi \sin\theta$ and $y = \rho \sin\phi \sin\theta$ and $z = \rho \cos\theta$. Solve these three equations for ρ, ϕ, and θ.

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2 + z^2} \\ \phi &= \tan^{-1} \frac{y}{x} \\ \theta &= \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} \end{aligned}$$

Use the chain rule to write $\partial G / \partial z$ in spherical coordinates.

$$\begin{aligned} \frac{\partial G}{\partial z} &= \frac{\partial G}{\partial \rho} \frac{\partial \rho}{\partial z} + \frac{\partial G}{\partial \phi} \frac{\partial \phi}{\partial z} + \frac{\partial G}{\partial \theta} \frac{\partial \theta}{\partial z} \\ &= \frac{\partial G}{\partial \rho} \left[\frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} (2z) \right] + \frac{\partial G}{\partial \phi} (0) + \frac{\partial G}{\partial \theta} \left[\frac{1}{1 + \left(\frac{\sqrt{x^2+y^2}}{z} \right)^2} \left(-\frac{\sqrt{x^2 + y^2}}{z^2} \right) \right] \\ &= \frac{\partial G}{\partial \rho} \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) - \frac{\partial G}{\partial \theta} \left(\frac{\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2} \right) \\ &= \frac{\partial G}{\partial \rho} (\cos\theta) - \frac{\partial G}{\partial \theta} \left(\frac{\sin\theta}{\rho} \right) \end{aligned}$$

$z = 0$ corresponds to $\theta = \pi/2$, so

$$\left. \frac{\partial G}{\partial z} \right|_{z=0} = \left[\frac{\partial G}{\partial \rho} (\cos\theta) - \frac{\partial G}{\partial \theta} \left(\frac{\sin\theta}{\rho} \right) \right] \bigg|_{\theta=\pi/2} = -\frac{1}{\rho} \left. \frac{\partial G}{\partial \theta} \right|_{\theta=\pi/2}.$$

As a result,

$$\begin{aligned} u(\rho_0,\phi_0,\theta_0) &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^R G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0)f(\rho,\phi,\theta)\rho^2\sin\theta\,d\rho\,d\phi\,d\theta \\ &\quad + R^2\int_0^{\pi/2}\int_0^{2\pi} F(\phi,\theta)\left.\frac{\partial G}{\partial \rho}\right|_{\rho=R}\sin\theta\,d\phi\,d\theta \\ &\quad - \int_0^{2\pi}\int_0^R H(\rho,\phi)\left(-\frac{1}{\rho}\left.\frac{\partial G}{\partial\theta}\right|_{\theta=\pi/2}\right)\rho\,d\rho\,d\phi. \end{aligned}$$

Switch the roles of ρ_0, ϕ_0, and θ_0 with those of ρ, ϕ, and θ, respectively.

$$\begin{aligned} u(\rho,\phi,\theta) &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^R G(\rho_0,\phi_0,\theta_0;\rho,\phi,\theta)f(\rho_0,\phi_0,\theta_0)\rho_0^2\sin\theta_0\,d\rho_0\,d\phi_0\,d\theta_0 \\ &\quad + R^2\int_0^{\pi/2}\int_0^{2\pi} F(\phi_0,\theta_0)\left.\frac{\partial G}{\partial \rho_0}\right|_{\rho_0=R}\sin\theta_0\,d\phi_0\,d\theta_0 \\ &\quad + \int_0^{2\pi}\int_0^R H(\rho_0,\phi_0)\left.\frac{\partial G}{\partial\theta_0}\right|_{\theta_0=\pi/2} d\rho_0\,d\phi_0 \end{aligned}$$

Therefore, using the fact that the Green's function is symmetric,

$$\boxed{\begin{aligned} u(\rho,\phi,\theta) &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^R G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0)f(\rho_0,\phi_0,\theta_0)\rho_0^2\sin\theta_0\,d\rho_0\,d\phi_0\,d\theta_0 \\ &\quad + R^2\int_0^{\pi/2}\int_0^{2\pi} F(\phi_0,\theta_0)\left.\frac{\partial G}{\partial \rho_0}\right|_{\rho_0=R}\sin\theta_0\,d\phi_0\,d\theta_0 \\ &\quad + \int_0^{2\pi}\int_0^R H(\rho_0,\phi_0)\left.\frac{\partial G}{\partial\theta_0}\right|_{\theta_0=\pi/2} d\rho_0\,d\phi_0. \end{aligned}}$$

The solution for Poisson's equation is known, then, if the Green's function inside the half-ball can be determined. Begin by finding the Green's function in infinite space (no boundaries).

$$\Delta g = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \qquad -\infty < x, y, z < \infty$$

g can be interpreted as the electrostatic potential, and $\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$ can be interpreted as the charge density for a unit charge located at (x_0, y_0, z_0). Since there are no boundaries, g is expected to vary solely as a function of the radial distance from (x_0, y_0, z_0): $g = g(\mathscr{r})$, where $\mathscr{r} = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$. Integrate both sides over a solid ball centered at (x_0, y_0, z_0) with radius $\mathscr{r}$.

$$\iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le \mathscr{r}^2}} \Delta g\,dV = \iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le \mathscr{r}^2}} \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\,dV$$

Since the ball contains (x_0, y_0, z_0), the right side is 1. Write the Laplacian operator Δ as ∇^2

$$\iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le \mathscr{r}^2}} \nabla^2 g\,dV = 1$$

and apply the divergence theorem.

$$\iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=\mathscr{r}^2}} \nabla g \cdot \hat{\boldsymbol{\mathscr{r}}}\, dS = 1$$

Here $\hat{\boldsymbol{\mathscr{r}}}$ is the unit vector normal to this ball at every point on the boundary.

$$\iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=\mathscr{r}^2}} \frac{dg}{d\mathscr{r}}\, dS = 1$$

Because g only depends on $\mathscr{r}$, its derivative is constant on the ball's boundary.

$$\frac{dg}{d\mathscr{r}} \iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=\mathscr{r}^2}} dS = 1$$

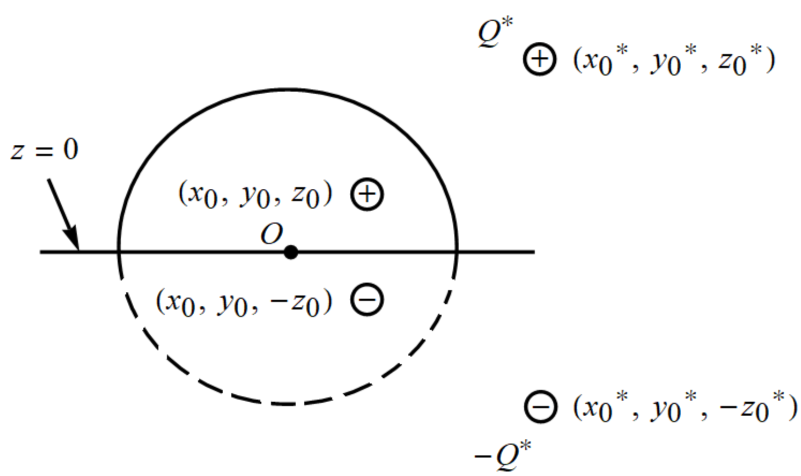
This surface integral is just the ball's surface area.

$$\frac{dg}{d\mathscr{r}}(4\pi\mathscr{r}^2) = 1 \quad \rightarrow \quad \frac{dg}{d\mathscr{r}} = \frac{1}{4\pi\mathscr{r}^2} \quad \rightarrow \quad g(\mathscr{r}) = -\frac{1}{4\pi\mathscr{r}}$$

The infinite-space Green's function is then

$$g(x, y, z; x_0, y_0, z_0) = -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}.$$

Now that it's known, the Green's function for the half-ball can be determined by the method of images. A convocation of point charges in infinite space will be arranged so that the boundary conditions, $G = 0$ along $\rho = R$ and $G = 0$ along $z = 0$, are satisfied.



For a positive unit charge located at (x_0, y_0, z_0) inside the half-ball, place a charge Q^* at (x_0^*, y_0^*, z_0^*) outside the half-ball such that the charges are collinear with the origin. Then place corresponding charges of opposite polarity at their reflections over the $z = 0$ plane. x_0^*, y_0^*, z_0^*, and Q^* are all unknown at the moment.

Write the Green's function in the half-ball (valid for $x^2 + y^2 + z^2 < R^2$, $z > 0$).

$$\begin{aligned}
G(x, y, z; x_0, y_0, z_0) &= +g(x, y, z; x_0, y_0, z_0) + Q^* g(x, y, z; x_0^*, y_0^*, z_0^*) \\
&\qquad - g(x, y, z; x_0, y_0, -z_0) - Q^* g(x, y, z; x_0^*, y_0^*, -z_0^*) \\
&= -\frac{1}{4\pi\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} - \frac{Q^*}{4\pi\sqrt{(x - x_0^*)^2 + (y - y_0^*)^2 + (z - z_0^*)^2}} \\
&\qquad + \frac{1}{4\pi\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2}} + \frac{Q^*}{4\pi\sqrt{(x - x_0^*)^2 + (y - y_0^*)^2 + (z + z_0^*)^2}} \\
&= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} + \frac{Q^*}{\sqrt{(x - x_0^*)^2 + (y - y_0^*)^2 + (z - z_0^*)^2}}\right] \\
&\qquad + \frac{1}{4\pi}\left[\frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2}} + \frac{Q^*}{\sqrt{(x - x_0^*)^2 + (y - y_0^*)^2 + (z + z_0^*)^2}}\right] \\
&= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x^2 + y^2 + z^2) + (x_0^2 + y_0^2 + z_0^2) - 2(xx_0 + yy_0 + zz_0)}}\right. \\
&\qquad\qquad \left. + \frac{Q^*}{\sqrt{(x^2 + y^2 + z^2) + (x_0^{*2} + y_0^{*2} + z_0^{*2}) - 2(xx_0^* + yy_0^* + zz_0^*)}}\right] \\
&\qquad + \frac{1}{4\pi}\left[\frac{1}{\sqrt{(x^2 + y^2 + z^2) + (x_0^2 + y_0^2 + z_0^2) - 2(xx_0 + yy_0 - zz_0)}}\right. \\
&\qquad\qquad \left. + \frac{Q^*}{\sqrt{(x^2 + y^2 + z^2) + (x_0^{*2} + y_0^{*2} + z_0^{*2}) - 2(xx_0^* + yy_0^* - zz_0^*)}}\right]
\end{aligned}$$

Change to spherical coordinates and simplify the formula.

$$\begin{aligned}
G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) &= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2[(\rho\cos\phi\sin\theta)(\rho_0\cos\phi_0\sin\theta_0)+(\rho\sin\phi\sin\theta)(\rho_0\sin\phi_0\sin\theta_0)+(\rho\cos\theta)(\rho_0\cos\theta_0)]}}\right.\\
&\qquad\left.+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2[(\rho\cos\phi\sin\theta)(\rho_0^*\cos\phi_0\sin\theta_0)+(\rho\sin\phi\sin\theta)(\rho_0^*\sin\phi_0\sin\theta_0)+(\rho\cos\theta)(\rho_0^*\cos\theta_0)]}}\right]\\
&\quad+\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2[(\rho\cos\phi\sin\theta)(\rho_0\cos\phi_0\sin\theta_0)+(\rho\sin\phi\sin\theta)(\rho_0\sin\phi_0\sin\theta_0)-(\rho\cos\theta)(\rho_0\cos\theta_0)]}}\right.\\
&\qquad\left.+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2[(\rho\cos\phi\sin\theta)(\rho_0^*\cos\phi_0\sin\theta_0)+(\rho\sin\phi\sin\theta)(\rho_0^*\sin\phi_0\sin\theta_0)-(\rho\cos\theta)(\rho_0^*\cos\theta_0)]}}\right]\\
&= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0(\cos\phi\cos\phi_0\sin\theta\sin\theta_0+\sin\phi\sin\phi_0\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)}}\right.\\
&\qquad\left.+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*(\cos\phi\cos\phi_0\sin\theta\sin\theta_0+\sin\phi\sin\phi_0\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)}}\right]\\
&\quad+\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0(\cos\phi\cos\phi_0\sin\theta\sin\theta_0+\sin\phi\sin\phi_0\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)}}\right.\\
&\qquad\left.+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*(\cos\phi\cos\phi_0\sin\theta\sin\theta_0+\sin\phi\sin\phi_0\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)}}\right]\\
&= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right.\\
&\qquad\left.+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]\\
&\quad+\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right.\\
&\qquad\left.+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right] \qquad (2)
\end{aligned}$$

Bring ρ out of the square root in the terms with 1 in the numerator, and bring ρ_0^* out of the square root in the terms with Q^* in the numerator.

$$\begin{aligned} G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) = -\frac{1}{4\pi}\Bigg[&\frac{1}{\rho\sqrt{1+\frac{\rho_0^2}{\rho^2}-\frac{2\rho_0}{\rho}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}} \\ &+\frac{Q^*}{\rho_0^*\sqrt{\frac{\rho^2}{\rho_0^{*2}}+1-\frac{2\rho}{\rho_0^*}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\Bigg] \\ +\frac{1}{4\pi}\Bigg[&\frac{1}{\rho\sqrt{1+\frac{\rho_0^2}{\rho^2}-\frac{2\rho_0}{\rho}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}} \\ &+\frac{Q^*}{\rho_0^*\sqrt{\frac{\rho^2}{\rho_0^{*2}}+1-\frac{2\rho}{\rho_0^*}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\Bigg] \end{aligned}$$

The potential at $\rho = R$ is zero.

$$\begin{aligned} G(R,\phi,\theta;\rho_0,\phi_0,\theta_0) = -\frac{1}{4\pi}\Bigg[&\frac{1}{R\sqrt{1+\frac{\rho_0^2}{R^2}-\frac{2\rho_0}{R}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}} \\ &+\frac{Q^*}{\rho_0^*\sqrt{1+\frac{R^2}{\rho_0^{*2}}-\frac{2R}{\rho_0^*}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\Bigg] \\ +\frac{1}{4\pi}\Bigg[&\frac{1}{R\sqrt{1+\frac{\rho_0^2}{R^2}-\frac{2\rho_0}{R}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}} \\ &+\frac{Q^*}{\rho_0^*\sqrt{1+\frac{R^2}{\rho_0^{*2}}-\frac{2R}{\rho_0^*}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\Bigg] = 0 \end{aligned}$$

In order for the quantities in square brackets to vanish, set

$$\frac{\rho_0}{R} = \frac{R}{\rho_0^*} \quad \text{and} \quad \frac{1}{R} + \frac{Q^*}{\rho_0^*} = 0,$$

which means

$$\begin{aligned} \rho_0^* = \frac{R^2}{\rho_0} \quad \text{and} \quad Q^* &= -\frac{\rho_0^*}{R} \\ Q^* &= -\frac{R}{\rho_0}. \end{aligned}$$

Consequently, the formula for the Green's function in equation (2) becomes

$$\begin{aligned}
G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) &= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right.\\
&\qquad\left.-\frac{\frac{R}{\rho_0}}{\sqrt{\rho^2+\frac{R^4}{\rho_0^2}-2\rho\frac{R^2}{\rho_0}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]\\
&\quad+\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right.\\
&\qquad\left.-\frac{\frac{R}{\rho_0}}{\sqrt{\rho^2+\frac{R^4}{\rho_0^2}-2\rho\frac{R^2}{\rho_0}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right]\\
&= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right.\\
&\qquad\left.-\frac{R}{\sqrt{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]\\
&\quad+\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right.\\
&\qquad\left.-\frac{R}{\sqrt{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right]\\
&= \frac{1}{4\pi}\left\{\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right.\\
&\qquad-\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\\
&\qquad+R\left[\frac{1}{\sqrt{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right.\\
&\qquad\left.\left.-\frac{1}{\sqrt{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right]\right\}.
\end{aligned}$$

Calculate the derivatives, $\partial G/\partial\rho_0$ and $\partial G/\partial\theta_0$, and evaluate them at $\rho_0 = R$ and $\theta_0 = \pi/2$, respectively.

$$\begin{aligned}
\left.\frac{\partial G}{\partial\rho_0}\right|_{\rho_0=R} &= \frac{R^2-\rho^2}{4\pi R}\left\{\frac{1}{\{R^2+\rho^2-2R\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]\}^{3/2}}\right.\\
&\qquad\left.-\frac{1}{\{R^2+\rho^2-2R\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]\}^{3/2}}\right\}\\
\left.\frac{\partial G}{\partial\theta_0}\right|_{\theta_0=\pi/2} &= \frac{\rho\rho_0\cos\theta}{2\pi}\left\{\frac{1}{\left\{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)\sin\theta\right\}^{3/2}}-\frac{R^3}{\left\{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0\cos(\phi-\phi_0)\sin\theta\right\}^{3/2}}\right\}
\end{aligned}$$