Exercise 14

Do the same for the eighth of a ball

$$D = \{x^2 + y^2 + z^2 < a^2,\ x > 0,\ y > 0,\ z > 0\}.$$

Solution

Here we will solve the Poisson equation inside an eighth-ball with radius R that is subject to boundary conditions on its spherical and planar surfaces. Use a spherical coordinate system (ρ, ϕ, θ) in which θ is the angle from the polar axis.

$$\begin{aligned}
&\Delta u = f(\rho, \phi, \theta), \quad \rho < R,\ 0 < \phi < \frac{\pi}{2},\ 0 < \theta < \frac{\pi}{2} \\
&u(R, \phi, \theta) = F(\phi, \theta) \\
&u\left(\rho, \phi, \frac{\pi}{2}\right) = H(\rho, \phi) \\
&u(\rho, 0, \theta) = A(\rho, \theta) \\
&u\left(\rho, \frac{\pi}{2}, \theta\right) = B(\rho, \theta)
\end{aligned}$$

A Green's function representation for the solution can be obtained from Green's second identity,

$$\iiint_D (u\Delta v - v\Delta u)\, dV = \iint_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS,$$

which holds for any two functions, u and v, over any domain and its boundary. Let v be the Green's function: $v = G = G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0)$.

$$\iiint_D (u\Delta G - G\Delta u)\, dV = \iint_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS \tag{1}$$

If we require it to satisfy

$$\begin{aligned}
&\Delta G = \delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(\theta - \theta_0), \qquad \rho < R,\ 0 < \phi < \frac{\pi}{2},\ 0 < \theta < \frac{\pi}{2} \\
&G = 0 \text{ on bdy } D,
\end{aligned}$$

where $(\rho_0, \phi_0, \theta_0)$ is a point in the eighth-ball, then equation (1) becomes

$$\begin{aligned}
\iiint_D [u(\rho, \phi, \theta)\delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(\theta - \theta_0) - G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho, \phi, \theta)]\, dV& \\
= \iint_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - (0)\frac{\partial u}{\partial n}\right) dS.&
\end{aligned}$$

Since the domain is an eighth-ball centered at the origin, there are four boundaries to consider. There's the spherical boundary $\rho = R$, where the outward unit normal vector is $\hat{\mathbf{n}} = \hat{\boldsymbol{\rho}}$ and the normal derivative is $\partial/\partial n = \partial/\partial\rho$. There are also the planar boundaries, $x = 0$, $y = 0$, and $z = 0$, where the outward unit normal vectors are $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$, $\hat{\mathbf{n}} = -\hat{\mathbf{y}}$, and $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$, respectively.

$$\iiint_D u(\rho, \phi, \theta)\delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(\theta - \theta_0)\, dV - \iiint_D G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho, \phi, \theta)\, dV = \iint_{\text{bdy } D} u\frac{\partial G}{\partial n}\, dS$$

The integral involving the delta functions is $u(\rho_0, \phi_0, \theta_0)$.

$$\begin{aligned}
u(\rho_0, \phi_0, \theta_0) &- \int_0^{\pi/2}\int_0^{\pi/2}\int_0^R G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) f(\rho,\phi,\theta)\rho^2\sin\theta\,d\rho\,d\phi\,d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2} u(R,\phi,\theta)\left.\frac{\partial G}{\partial\rho}\right|_{\rho=R} R^2\sin\theta\,d\phi\,d\theta + \int_0^{\pi/2}\int_0^R u\left(\rho,\phi,\frac{\pi}{2}\right)\left.\left(-\frac{\partial G}{\partial z}\right)\right|_{z=0}\rho\,d\rho\,d\phi \\
&\quad + \int_0^{\pi/2}\int_0^R u\left(\rho,\frac{\pi}{2},\theta\right)\left.\left(-\frac{\partial G}{\partial x}\right)\right|_{x=0}\rho\,d\rho\,d\theta + \int_0^{\pi/2}\int_0^R u(\rho,0,\theta)\left.\left(-\frac{\partial G}{\partial y}\right)\right|_{y=0}\rho\,d\rho\,d\theta
\end{aligned}$$

Solve for u.

$$\begin{aligned}
u(\rho_0, \phi_0, \theta_0) = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^R & G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) f(\rho,\phi,\theta)\rho^2\sin\theta\,d\rho\,d\phi\,d\theta \\
&+ R^2\int_0^{\pi/2}\int_0^{\pi/2} F(\phi,\theta)\left.\frac{\partial G}{\partial\rho}\right|_{\rho=R}\sin\theta\,d\phi\,d\theta - \int_0^{\pi/2}\int_0^R H(\rho,\phi)\left.\frac{\partial G}{\partial z}\right|_{z=0}\rho\,d\rho\,d\phi \\
&- \int_0^{\pi/2}\int_0^R B(\rho,\theta)\left.\frac{\partial G}{\partial x}\right|_{x=0}\rho\,d\rho\,d\theta - \int_0^{\pi/2}\int_0^R A(\rho,\theta)\left.\frac{\partial G}{\partial y}\right|_{y=0}\rho\,d\rho\,d\theta
\end{aligned}$$

In spherical coordinates, $x = \rho\cos\phi\sin\theta$ and $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\theta$. Solve these three equations for ρ, ϕ, and θ.

$$\begin{aligned}
\rho &= \sqrt{x^2+y^2+z^2} \\
\phi &= \tan^{-1}\frac{y}{x} \\
\theta &= \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}
\end{aligned}$$

Use the chain rule to write $\partial G/\partial x$ in spherical coordinates.

$$\begin{aligned}
\frac{\partial G}{\partial x} &= \frac{\partial G}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial G}{\partial\phi}\frac{\partial\phi}{\partial x} + \frac{\partial G}{\partial\theta}\frac{\partial\theta}{\partial x} \\
&= \frac{\partial G}{\partial\rho}\left[\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)\right] + \frac{\partial G}{\partial\phi}\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)\right] + \frac{\partial G}{\partial\theta}\left[\frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\left(\frac{1}{2z\sqrt{x^2+y^2}}\cdot 2x\right)\right] \\
&= \frac{\partial G}{\partial\rho}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial G}{\partial\phi}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial G}{\partial\theta}\left(\frac{z}{x^2+y^2+z^2}\frac{x}{\sqrt{x^2+y^2}}\right) \\
&= \frac{\partial G}{\partial\rho}(\cos\phi\sin\theta) - \frac{\partial G}{\partial\phi}\left(\frac{\sin\phi}{\rho\sin\theta}\right) + \frac{\partial G}{\partial\theta}\left(\frac{\cos\theta}{\rho}\cos\phi\right)
\end{aligned}$$

$x = 0$ corresponds to $\phi = \pi/2$, so

$$\left.\frac{\partial G}{\partial x}\right|_{x=0} = \left.\left[\frac{\partial G}{\partial\rho}(\cos\phi\sin\theta) - \frac{\partial G}{\partial\phi}\left(\frac{\sin\phi}{\rho\sin\theta}\right) + \frac{\partial G}{\partial\theta}\left(\frac{\cos\theta}{\rho}\cos\phi\right)\right]\right|_{\phi=\pi/2} = -\frac{1}{\rho\sin\theta}\left.\frac{\partial G}{\partial\phi}\right|_{\phi=\pi/2}.$$

Use the chain rule to write $\partial G/\partial y$ in spherical coordinates.

$$\begin{aligned}
\frac{\partial G}{\partial y} &= \frac{\partial G}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial G}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial G}{\partial \theta}\frac{\partial \theta}{\partial y} \\
&= \frac{\partial G}{\partial \rho}\left[\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)\right] + \frac{\partial G}{\partial \phi}\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)\right] + \frac{\partial G}{\partial \theta}\left[\frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\left(\frac{1}{2z\sqrt{x^2+y^2}}\cdot 2y\right)\right] \\
&= \frac{\partial G}{\partial \rho}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial G}{\partial \phi}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial G}{\partial \theta}\left(\frac{z}{x^2+y^2+z^2}\frac{y}{\sqrt{x^2+y^2}}\right) \\
&= \frac{\partial G}{\partial \rho}(\sin\phi\sin\theta) + \frac{\partial G}{\partial \phi}\left(\frac{\cos\phi}{\rho\sin\theta}\right) + \frac{\partial G}{\partial \theta}\left(\frac{\cos\theta}{\rho}\sin\phi\right)
\end{aligned}$$

$y=0$ corresponds to $\phi=0$, so

$$\left.\frac{\partial G}{\partial y}\right|_{y=0} = \left.\left[\frac{\partial G}{\partial \rho}(\sin\phi\sin\theta) + \frac{\partial G}{\partial \phi}\left(\frac{\cos\phi}{\rho\sin\theta}\right) + \frac{\partial G}{\partial \theta}\left(\frac{\cos\theta}{\rho}\sin\phi\right)\right]\right|_{\phi=0} = \frac{1}{\rho\sin\theta}\left.\frac{\partial G}{\partial \phi}\right|_{\phi=0}.$$

Use the chain rule to write $\partial G/\partial z$ in spherical coordinates.

$$\begin{aligned}
\frac{\partial G}{\partial z} &= \frac{\partial G}{\partial \rho}\frac{\partial \rho}{\partial z} + \frac{\partial G}{\partial \phi}\frac{\partial \phi}{\partial z} + \frac{\partial G}{\partial \theta}\frac{\partial \theta}{\partial z} \\
&= \frac{\partial G}{\partial \rho}\left[\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)\right] + \frac{\partial G}{\partial \phi}(0) + \frac{\partial G}{\partial \theta}\left[\frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\left(-\frac{\sqrt{x^2+y^2}}{z^2}\right)\right] \\
&= \frac{\partial G}{\partial \rho}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial G}{\partial \theta}\left(\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}\right) \\
&= \frac{\partial G}{\partial \rho}(\cos\theta) - \frac{\partial G}{\partial \theta}\left(\frac{\sin\theta}{\rho}\right)
\end{aligned}$$

$z=0$ corresponds to $\theta=\pi/2$, so

$$\left.\frac{\partial G}{\partial z}\right|_{z=0} = \left.\left[\frac{\partial G}{\partial \rho}(\cos\theta) - \frac{\partial G}{\partial \theta}\left(\frac{\sin\theta}{\rho}\right)\right]\right|_{\theta=\pi/2} = -\frac{1}{\rho}\left.\frac{\partial G}{\partial \theta}\right|_{\theta=\pi/2}.$$

As a result,

$$\begin{aligned}
u(\rho_0,\phi_0,\theta_0) &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^R G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0)f(\rho,\phi,\theta)\rho^2\sin\theta\,d\rho\,d\phi\,d\theta \\
&\quad + R^2\int_0^{\pi/2}\int_0^{\pi/2} F(\phi,\theta)\left.\frac{\partial G}{\partial \rho}\right|_{\rho=R}\sin\theta\,d\phi\,d\theta - \int_0^{\pi/2}\int_0^R H(\rho,\phi)\left(-\frac{1}{\rho}\left.\frac{\partial G}{\partial \theta}\right|_{\theta=\pi/2}\right)\rho\,d\rho\,d\phi \\
&\quad - \int_0^{\pi/2}\int_0^R B(\rho,\theta)\left(-\frac{1}{\rho\sin\theta}\left.\frac{\partial G}{\partial \phi}\right|_{\phi=\pi/2}\right)\rho\,d\rho\,d\theta - \int_0^{\pi/2}\int_0^R A(\rho,\theta)\left(\frac{1}{\rho\sin\theta}\left.\frac{\partial G}{\partial \phi}\right|_{\phi=0}\right)\rho\,d\rho\,d\theta.
\end{aligned}$$

Switch the roles of ρ_0, ϕ_0, and θ_0 with those of ρ, ϕ, and θ, respectively.

$$\begin{aligned}u(\rho,\phi,\theta) &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^R G(\rho_0,\phi_0,\theta_0;\rho,\phi,\theta)f(\rho_0,\phi_0,\theta_0)\rho_0^2\sin\theta_0\,d\rho_0\,d\phi_0\,d\theta_0 \\ &\quad + R^2\int_0^{\pi/2}\int_0^{\pi/2} F(\phi_0,\theta_0)\frac{\partial G}{\partial\rho_0}\bigg|_{\rho_0=R}\sin\theta_0\,d\phi_0\,d\theta_0 + \int_0^{\pi/2}\int_0^R H(\rho_0,\phi_0)\frac{\partial G}{\partial\theta_0}\bigg|_{\theta_0=\pi/2}d\rho_0\,d\phi_0 \\ &\quad + \int_0^{\pi/2}\int_0^R \frac{B(\rho_0,\theta_0)}{\sin\theta_0}\frac{\partial G}{\partial\phi_0}\bigg|_{\phi_0=\pi/2}d\rho_0\,d\theta_0 - \int_0^{\pi/2}\int_0^R \frac{A(\rho_0,\theta_0)}{\sin\theta_0}\frac{\partial G}{\partial\phi_0}\bigg|_{\phi_0=0}d\rho_0\,d\theta_0.\end{aligned}$$

Therefore, using the fact that the Green's function is symmetric,

$$\begin{aligned}u(\rho,\phi,\theta) &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^R G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0)f(\rho_0,\phi_0,\theta_0)\rho_0^2\sin\theta_0\,d\rho_0\,d\phi_0\,d\theta_0 \\ &\quad + R^2\int_0^{\pi/2}\int_0^{\pi/2} F(\phi_0,\theta_0)\frac{\partial G}{\partial\rho_0}\bigg|_{\rho_0=R}\sin\theta_0\,d\phi_0\,d\theta_0 + \int_0^{\pi/2}\int_0^R H(\rho_0,\phi_0)\frac{\partial G}{\partial\theta_0}\bigg|_{\theta_0=\pi/2}d\rho_0\,d\phi_0 \\ &\quad + \int_0^{\pi/2}\int_0^R \frac{B(\rho_0,\theta_0)}{\sin\theta_0}\frac{\partial G}{\partial\phi_0}\bigg|_{\phi_0=\pi/2}d\rho_0\,d\theta_0 - \int_0^{\pi/2}\int_0^R \frac{A(\rho_0,\theta_0)}{\sin\theta_0}\frac{\partial G}{\partial\phi_0}\bigg|_{\phi_0=0}d\rho_0\,d\theta_0.\end{aligned}$$

The solution for Poisson's equation is known, then, if the Green's function inside the eighth-ball can be determined. Begin by finding the Green's function in infinite space (no boundaries).

$$\Delta g = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \qquad -\infty < x, y, z < \infty$$

g can be interpreted as the electrostatic potential, and $\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$ can be interpreted as the charge density for a unit charge located at (x_0, y_0, z_0). Since there are no boundaries, g is expected to vary solely as a function of the radial distance from (x_0, y_0, z_0): $g = g(\imath)$, where $\imath = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$. Integrate both sides over a solid ball centered at (x_0, y_0, z_0) with radius $\imath$.

$$\iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le\imath^2}} \Delta g\,dV = \iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le\imath^2}} \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\,dV$$

Since the ball contains (x_0, y_0, z_0), the right side is 1. Write the Laplacian operator Δ as ∇^2

$$\iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le\imath^2}} \nabla^2 g\,dV = 1$$

$$\iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le\imath^2}} \nabla\cdot\nabla g\,dV = 1$$

and apply the divergence theorem.

$$\iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=\imath^2}} \nabla g\cdot\hat{\imath}\,dS = 1$$

Here $\hat{\boldsymbol{r}}$ is the unit vector normal to this ball at every point on the boundary.

$$\iint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=r^2}} \frac{dg}{dr}\,dS = 1$$

Because g only depends on r, its derivative is constant on the ball's boundary.

$$\frac{dg}{dr}\iint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=r^2}} dS = 1$$

This surface integral is just the ball's surface area.

$$\frac{dg}{dr}(4\pi r^2) = 1$$

Divide both sides by $4\pi r^2$.

$$\frac{dg}{dr} = \frac{1}{4\pi r^2}$$

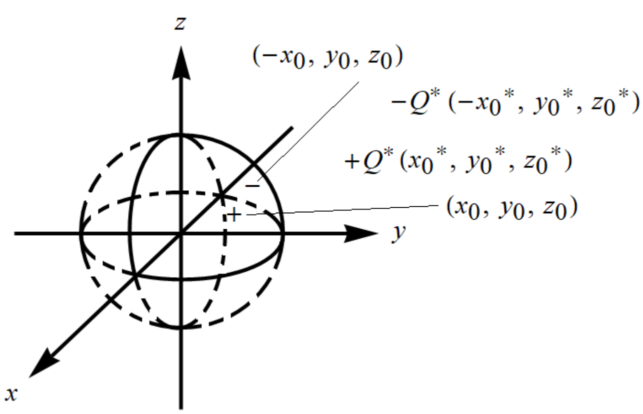
Integrate both sides with respect to r.

$$g(r) = -\frac{1}{4\pi r}$$

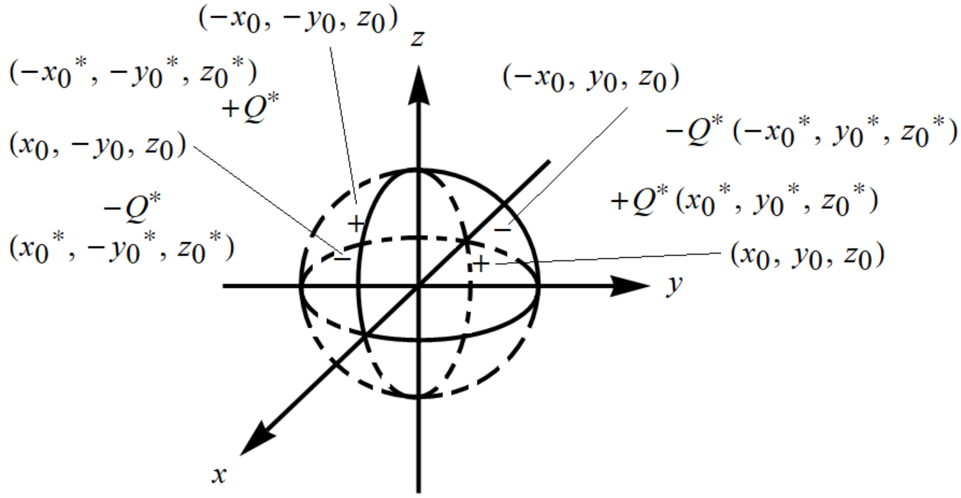
The infinite-space Green's function is then

$$g(x, y, z; x_0, y_0, z_0) = -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}.$$

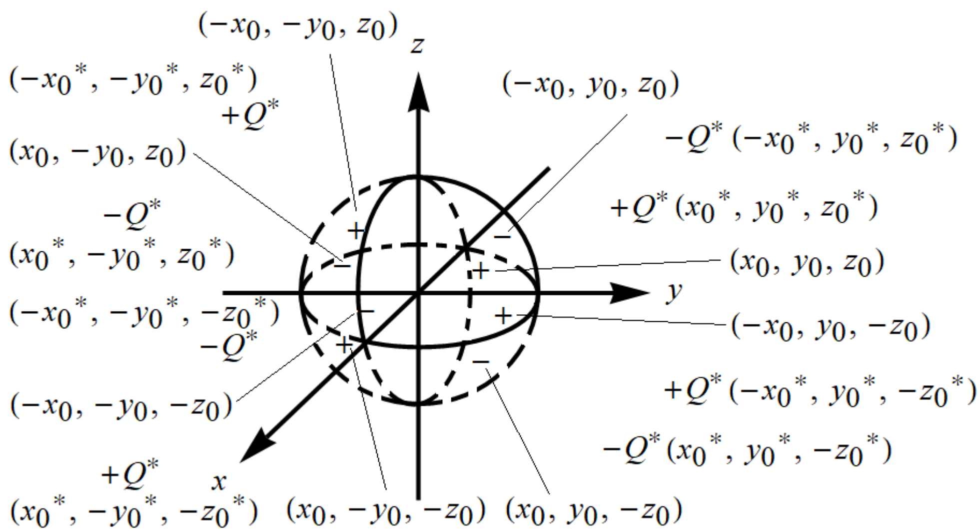
Now that it's known, the Green's function for the eighth-ball can be determined by the method of images. A convocation of point charges in infinite space will be arranged so that the boundary conditions, $G = 0$ along $\rho = R$ and $G = 0$ along $x = 0$, $y = 0$, and $z = 0$, are satisfied. For a positive unit charge located at (x_0, y_0, z_0) inside the eighth-ball, place a charge Q^* at (x_0^*, y_0^*, z_0^*) outside the eighth-ball such that the charges are collinear with the origin. Then place two corresponding charges of opposite polarity at their reflections over the yz-plane so that $G = 0$ on $x = 0$ is satisfied.



Then place four corresponding charges of opposite polarity at their reflections over the xz-plane so that $G = 0$ on $y = 0$ is satisfied.



Then place eight corresponding charges of opposite polarity at their reflections over the xy-plane so that $G = 0$ on $z = 0$ is satisfied.



x_0^*, y_0^*, z_0^*, and Q^* are all unknown at the moment.

Write the Green's function in the eighth-ball (valid for $x^2+y^2+z^2<R^2$, $x>0$, $y>0$, $z>0$).

$$\begin{aligned}
G(x,y,z;x_0,y_0,z_0) &= +g(x,y,z;x_0,y_0,z_0) + Q^*g(x,y,z;x_0^*,y_0^*,z_0^*) \\
&\quad - g(x,y,z;-x_0,y_0,z_0) - Q^*g(x,y,z;-x_0^*,y_0^*,z_0^*) \\
&\quad + g(x,y,z;-x_0,y_0,-z_0) + Q^*g(x,y,z;-x_0^*,y_0^*,-z_0^*) \\
&\quad - g(x,y,z;x_0,y_0,-z_0) - Q^*g(x,y,z;x_0^*,y_0^*,-z_0^*) \\
&\quad + g(x,y,z;-x_0,-y_0,z_0) + Q^*g(x,y,z;-x_0^*,-y_0^*,z_0^*) \\
&\quad - g(x,y,z;x_0,-y_0,z_0) - Q^*g(x,y,z;x_0^*,-y_0^*,z_0^*) \\
&\quad + g(x,y,z;x_0,-y_0,-z_0) + Q^*g(x,y,z;x_0^*,-y_0^*,-z_0^*) \\
&\quad - g(x,y,z;-x_0,-y_0,-z_0) - Q^*g(x,y,z;-x_0^*,-y_0^*,-z_0^*) \\
&= -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} - \frac{Q^*}{4\pi\sqrt{(x-x_0^*)^2+(y-y_0^*)^2+(z-z_0^*)^2}} \\
&\quad + \frac{1}{4\pi\sqrt{(x+x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \frac{Q^*}{4\pi\sqrt{(x+x_0^*)^2+(y-y_0^*)^2+(z-z_0^*)^2}} \\
&\quad - \frac{1}{4\pi\sqrt{(x+x_0)^2+(y-y_0)^2+(z+z_0)^2}} - \frac{Q^*}{4\pi\sqrt{(x+x_0^*)^2+(y-y_0^*)^2+(z+z_0^*)^2}} \\
&\quad + \frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}} + \frac{Q^*}{4\pi\sqrt{(x-x_0^*)^2+(y-y_0^*)^2+(z+z_0^*)^2}} \\
&\quad - \frac{1}{4\pi\sqrt{(x+x_0)^2+(y+y_0)^2+(z-z_0)^2}} - \frac{Q^*}{4\pi\sqrt{(x+x_0^*)^2+(y+y_0^*)^2+(z-z_0^*)^2}} \\
&\quad + \frac{1}{4\pi\sqrt{(x-x_0)^2+(y+y_0)^2+(z-z_0)^2}} + \frac{Q^*}{4\pi\sqrt{(x-x_0^*)^2+(y+y_0^*)^2+(z-z_0^*)^2}} \\
&\quad - \frac{1}{4\pi\sqrt{(x-x_0)^2+(y+y_0)^2+(z+z_0)^2}} - \frac{Q^*}{4\pi\sqrt{(x-x_0^*)^2+(y+y_0^*)^2+(z+z_0^*)^2}} \\
&\quad + \frac{1}{4\pi\sqrt{(x+x_0)^2+(y+y_0)^2+(z+z_0)^2}} + \frac{Q^*}{4\pi\sqrt{(x+x_0^*)^2+(y+y_0^*)^2+(z+z_0^*)^2}}
\end{aligned}$$

Factor each pair of terms and expand the denominators.

$$\begin{aligned}
G(x,y,z;x_0,y_0,z_0) = &-\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x^2+y^2+z^2)+(x_0^2+y_0^2+z_0^2)-2(xx_0+yy_0+zz_0)}}+\frac{Q^*}{\sqrt{(x^2+y^2+z^2)+(x_0^{*2}+y_0^{*2}+z_0^{*2})-2(xx_0^*+yy_0^*+zz_0^*)}}\right] \\
&+\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x^2+y^2+z^2)+(x_0^2+y_0^2+z_0^2)-2(-xx_0+yy_0+zz_0)}}+\frac{Q^*}{\sqrt{(x^2+y^2+z^2)+(x_0^{*2}+y_0^{*2}+z_0^{*2})-2(-xx_0^*+yy_0^*+zz_0^*)}}\right] \\
&-\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x^2+y^2+z^2)+(x_0^2+y_0^2+z_0^2)-2(-xx_0+yy_0-zz_0)}}+\frac{Q^*}{\sqrt{(x^2+y^2+z^2)+(x_0^{*2}+y_0^{*2}+z_0^{*2})-2(-xx_0^*+yy_0^*-zz_0^*)}}\right] \\
&+\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x^2+y^2+z^2)+(x_0^2+y_0^2+z_0^2)-2(xx_0+yy_0-zz_0)}}+\frac{Q^*}{\sqrt{(x^2+y^2+z^2)+(x_0^{*2}+y_0^{*2}+z_0^{*2})-2(xx_0^*+yy_0^*-zz_0^*)}}\right] \\
&-\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x^2+y^2+z^2)+(x_0^2+y_0^2+z_0^2)-2(-xx_0-yy_0+zz_0)}}+\frac{Q^*}{\sqrt{(x^2+y^2+z^2)+(x_0^{*2}+y_0^{*2}+z_0^{*2})-2(-xx_0^*-yy_0^*+zz_0^*)}}\right] \\
&+\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x^2+y^2+z^2)+(x_0^2+y_0^2+z_0^2)-2(xx_0-yy_0+zz_0)}}+\frac{Q^*}{\sqrt{(x^2+y^2+z^2)+(x_0^{*2}+y_0^{*2}+z_0^{*2})-2(xx_0^*-yy_0^*+zz_0^*)}}\right] \\
&-\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x^2+y^2+z^2)+(x_0^2+y_0^2+z_0^2)-2(xx_0-yy_0-zz_0)}}+\frac{Q^*}{\sqrt{(x^2+y^2+z^2)+(x_0^{*2}+y_0^{*2}+z_0^{*2})-2(xx_0^*-yy_0^*-zz_0^*)}}\right] \\
&+\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x^2+y^2+z^2)+(x_0^2+y_0^2+z_0^2)-2(-xx_0-yy_0-zz_0)}}+\frac{Q^*}{\sqrt{(x^2+y^2+z^2)+(x_0^{*2}+y_0^{*2}+z_0^{*2})-2(-xx_0^*-yy_0^*-zz_0^*)}}\right]
\end{aligned}$$

Change to spherical coordinates and simplify the formula.

$$
\begin{aligned}
G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) = -\frac{1}{4\pi}&\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right] \\
+\frac{1}{4\pi}&\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right] \\
-\frac{1}{4\pi}&\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right] \\
+\frac{1}{4\pi}&\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right] \\
-\frac{1}{4\pi}&\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right] \\
+\frac{1}{4\pi}&\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*[\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right] \\
-\frac{1}{4\pi}&\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*[\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right] \\
+\frac{1}{4\pi}&\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right]
\end{aligned}
$$

Bring ρ out of the square root in the terms with 1 in the numerator, and bring ρ_0^* out of the square root in the terms with Q^* in the numerator.

$$\begin{aligned}
G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) = -\frac{1}{4\pi}&\left[\frac{1}{\rho\sqrt{1+\frac{\rho_0^2}{\rho^2}-\frac{2\rho_0}{\rho}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{\frac{\rho^2}{\rho_0^{*2}}+1-\frac{2\rho}{\rho_0^*}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right] \\
+\frac{1}{4\pi}&\left[\frac{1}{\rho\sqrt{1+\frac{\rho_0^2}{\rho^2}-\frac{2\rho_0}{\rho}[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{\frac{\rho^2}{\rho_0^{*2}}+1-\frac{2\rho}{\rho_0^*}[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right] \\
-\frac{1}{4\pi}&\left[\frac{1}{\rho\sqrt{1+\frac{\rho_0^2}{\rho^2}-\frac{2\rho_0}{\rho}[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{\frac{\rho^2}{\rho_0^{*2}}+1-\frac{2\rho}{\rho_0^*}[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right] \\
+\frac{1}{4\pi}&\left[\frac{1}{\rho\sqrt{1+\frac{\rho_0^2}{\rho^2}-\frac{2\rho_0}{\rho}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{\frac{\rho^2}{\rho_0^{*2}}+1-\frac{2\rho}{\rho_0^*}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right] \\
-\frac{1}{4\pi}&\left[\frac{1}{\rho\sqrt{1+\frac{\rho_0^2}{\rho^2}-\frac{2\rho_0}{\rho}[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{\frac{\rho^2}{\rho_0^{*2}}+1-\frac{2\rho}{\rho_0^*}[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right] \\
+\frac{1}{4\pi}&\left[\frac{1}{\rho\sqrt{1+\frac{\rho_0^2}{\rho^2}-\frac{2\rho_0}{\rho}[\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{\frac{\rho^2}{\rho_0^{*2}}+1-\frac{2\rho}{\rho_0^*}[\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right] \\
-\frac{1}{4\pi}&\left[\frac{1}{\rho\sqrt{1+\frac{\rho_0^2}{\rho^2}-\frac{2\rho_0}{\rho}[\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{\frac{\rho^2}{\rho_0^{*2}}+1-\frac{2\rho}{\rho_0^*}[\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right] \\
+\frac{1}{4\pi}&\left[\frac{1}{\rho\sqrt{1+\frac{\rho_0^2}{\rho^2}-\frac{2\rho_0}{\rho}[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{\frac{\rho^2}{\rho_0^{*2}}+1-\frac{2\rho}{\rho_0^*}[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right]
\end{aligned}$$

The potential at $\rho = R$ is zero.

$$\begin{aligned}
0 = -\frac{1}{4\pi}&\left[\frac{1}{R\sqrt{1+\frac{\rho_0^2}{R^2}-\frac{2\rho_0}{R}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{1+\frac{R^2}{\rho_0^{*2}}-\frac{2R}{\rho_0^*}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right] \\
+\frac{1}{4\pi}&\left[\frac{1}{R\sqrt{1+\frac{\rho_0^2}{R^2}-\frac{2\rho_0}{R}[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{1+\frac{R^2}{\rho_0^{*2}}-\frac{2R}{\rho_0^*}[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right] \\
-\frac{1}{4\pi}&\left[\frac{1}{R\sqrt{1+\frac{\rho_0^2}{R^2}-\frac{2\rho_0}{R}[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{1+\frac{R^2}{\rho_0^{*2}}-\frac{2R}{\rho_0^*}[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right] \\
+\frac{1}{4\pi}&\left[\frac{1}{R\sqrt{1+\frac{\rho_0^2}{R^2}-\frac{2\rho_0}{R}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{1+\frac{R^2}{\rho_0^{*2}}-\frac{2R}{\rho_0^*}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right] \\
-\frac{1}{4\pi}&\left[\frac{1}{R\sqrt{1+\frac{\rho_0^2}{R^2}-\frac{2\rho_0}{R}[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{1+\frac{R^2}{\rho_0^{*2}}-\frac{2R}{\rho_0^*}[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right] \\
+\frac{1}{4\pi}&\left[\frac{1}{R\sqrt{1+\frac{\rho_0^2}{R^2}-\frac{2\rho_0}{R}[\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{1+\frac{R^2}{\rho_0^{*2}}-\frac{2R}{\rho_0^*}[\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right] \\
-\frac{1}{4\pi}&\left[\frac{1}{R\sqrt{1+\frac{\rho_0^2}{R^2}-\frac{2\rho_0}{R}[\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{1+\frac{R^2}{\rho_0^{*2}}-\frac{2R}{\rho_0^*}[\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right] \\
+\frac{1}{4\pi}&\left[\frac{1}{R\sqrt{1+\frac{\rho_0^2}{R^2}-\frac{2\rho_0}{R}[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{Q^*}{\rho_0^*\sqrt{1+\frac{R^2}{\rho_0^{*2}}-\frac{2R}{\rho_0^*}[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right]
\end{aligned}$$

In order for the quantities in square brackets to vanish, set

$$\frac{\rho_0}{R} = \frac{R}{\rho_0^*} \quad \text{and} \quad \frac{1}{R} + \frac{Q^*}{\rho_0^*} = 0 \quad \rightarrow \quad \rho_0^* = \frac{R^2}{\rho_0} \quad \text{and} \quad Q^* = -\frac{\rho_0^*}{R} = -\frac{R}{\rho_0}.$$

With these values for ρ_0^* and Q^*, the first equation for G in spherical coordinates becomes

$$\begin{aligned}
G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) = &-\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{-\frac{R}{\rho_0}}{\sqrt{\rho^2+\frac{R^4}{\rho_0^2}-2\rho\frac{R^2}{\rho_0}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]\\
&+\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{-\frac{R}{\rho_0}}{\sqrt{\rho^2+\frac{R^4}{\rho_0^2}-2\rho\frac{R^2}{\rho_0}[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]\\
&-\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{-\frac{R}{\rho_0}}{\sqrt{\rho^2+\frac{R^4}{\rho_0^2}-2\rho\frac{R^2}{\rho_0}[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right]\\
&+\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{-\frac{R}{\rho_0}}{\sqrt{\rho^2+\frac{R^4}{\rho_0^2}-2\rho\frac{R^2}{\rho_0}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right]\\
&-\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{-\frac{R}{\rho_0}}{\sqrt{\rho^2+\frac{R^4}{\rho_0^2}-2\rho\frac{R^2}{\rho_0}[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]\\
&+\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}+\frac{-\frac{R}{\rho_0}}{\sqrt{\rho^2+\frac{R^4}{\rho_0^2}-2\rho\frac{R^2}{\rho_0}[\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]\\
&-\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{-\frac{R}{\rho_0}}{\sqrt{\rho^2+\frac{R^4}{\rho_0^2}-2\rho\frac{R^2}{\rho_0}[\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right]\\
&+\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}+\frac{-\frac{R}{\rho_0}}{\sqrt{\rho^2+\frac{R^4}{\rho_0^2}-2\rho\frac{R^2}{\rho_0}[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right].
\end{aligned}$$

Therefore, the Green's function in the eighth-ball is

$$\begin{aligned}
G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) = &-\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}-\frac{R}{\sqrt{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]\\
&+\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}-\frac{R}{\sqrt{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]\\
&-\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}-\frac{R}{\sqrt{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right]\\
&+\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}-\frac{R}{\sqrt{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right]\\
&-\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}-\frac{R}{\sqrt{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]\\
&+\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}-\frac{R}{\sqrt{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0[\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]\\
&-\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}-\frac{R}{\sqrt{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0[\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right]\\
&+\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}-\frac{R}{\sqrt{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]}}\right].
\end{aligned}$$

Calculate $\partial G/\partial \rho_0$ and evaluate it at $\rho_0 = R$.

$$\begin{aligned}
\left.\frac{\partial G}{\partial \rho_0}\right|_{\rho_0=R} = \frac{R^2-\rho^2}{4\pi R} & \frac{1}{\{R^2+\rho^2-2R\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]\}^{3/2}} \\
& - \frac{R^2-\rho^2}{4\pi R}\frac{1}{\{R^2+\rho^2-2R\rho[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]\}^{3/2}} \\
& + \frac{R^2-\rho^2}{4\pi R}\frac{1}{\{R^2+\rho^2-2R\rho[-\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]\}^{3/2}} \\
& - \frac{R^2-\rho^2}{4\pi R}\frac{1}{\{R^2+\rho^2-2R\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]\}^{3/2}} \\
& + \frac{R^2-\rho^2}{4\pi R}\frac{1}{\{R^2+\rho^2-2R\rho[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]\}^{3/2}} \\
& - \frac{R^2-\rho^2}{4\pi R}\frac{1}{\{R^2+\rho^2-2R\rho[\cos(\phi+\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]\}^{3/2}} \\
& + \frac{R^2-\rho^2}{4\pi R}\frac{1}{\{R^2+\rho^2-2R\rho[\cos(\phi+\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]\}^{3/2}} \\
& - \frac{R^2-\rho^2}{4\pi R}\frac{1}{\{R^2+\rho^2-2R\rho[-\cos(\phi-\phi_0)\sin\theta\sin\theta_0-\cos\theta\cos\theta_0]\}^{3/2}}
\end{aligned}$$

Calculate $\partial G/\partial\phi_0$ and evaluate it at $\phi_0 = 0$.

$$\begin{aligned}
\left.\frac{\partial G}{\partial \phi_0}\right|_{\phi_0=0} = &-\frac{1}{4\pi}\left[\frac{\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(\cos\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(\cos\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&+\frac{1}{4\pi}\left[\frac{\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(-\cos\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(-\cos\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&-\frac{1}{4\pi}\left[\frac{\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(-\cos\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(-\cos\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&+\frac{1}{4\pi}\left[\frac{\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(\cos\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(\cos\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&-\frac{1}{4\pi}\left[\frac{-\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(-\cos\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}} + \frac{R^3\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(-\cos\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&+\frac{1}{4\pi}\left[\frac{-\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(\cos\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}} + \frac{R^3\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(\cos\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&-\frac{1}{4\pi}\left[\frac{-\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(\cos\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}} + \frac{R^3\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(\cos\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&+\frac{1}{4\pi}\left[\frac{-\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(-\cos\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}} + \frac{R^3\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(-\cos\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}}\right] \\
= &-\frac{1}{2\pi}\left[\frac{\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(\cos\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(\cos\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&+\frac{1}{2\pi}\left[\frac{\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(-\cos\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(-\cos\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&-\frac{1}{2\pi}\left[\frac{\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(-\cos\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(-\cos\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&+\frac{1}{2\pi}\left[\frac{\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(\cos\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\sin\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(\cos\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}}\right]
\end{aligned}$$

Calculate $\partial G/\partial\phi_0$ and evaluate it at $\phi_0 = \pi/2$.

$$
\begin{aligned}
\left.\frac{\partial G}{\partial \phi_0}\right|_{\phi_0=\pi/2} = &-\frac{1}{4\pi}\left[\frac{-\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(\sin\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}} + \frac{R^3\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(\sin\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&+\frac{1}{4\pi}\left[\frac{\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(\sin\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(\sin\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&-\frac{1}{4\pi}\left[\frac{\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(\sin\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(\sin\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&+\frac{1}{4\pi}\left[\frac{-\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(\sin\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}} + \frac{R^3\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(\sin\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&-\frac{1}{4\pi}\left[\frac{\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(-\sin\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(-\sin\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&+\frac{1}{4\pi}\left[\frac{-\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(-\sin\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}} + \frac{R^3\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(-\sin\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&-\frac{1}{4\pi}\left[\frac{-\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(-\sin\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}} + \frac{R^3\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(-\sin\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&+\frac{1}{4\pi}\left[\frac{\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(-\sin\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(-\sin\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}}\right] \\
= &\frac{1}{2\pi}\left[\frac{\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(\sin\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(\sin\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&-\frac{1}{2\pi}\left[\frac{\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(\sin\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(\sin\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&-\frac{1}{2\pi}\left[\frac{\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(-\sin\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(-\sin\phi\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)]^{3/2}}\right] \\
&+\frac{1}{2\pi}\left[\frac{\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2+\rho_0^2-2\rho\rho_0(-\sin\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}} - \frac{R^3\rho\rho_0\cos\phi\sin\theta\sin\theta_0}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0(-\sin\phi\sin\theta\sin\theta_0-\cos\theta\cos\theta_0)]^{3/2}}\right]
\end{aligned}
$$

Finally, calculate $\partial G/\partial\theta_0$ and evaluate it at $\theta_0 = \pi/2$.

$$\begin{aligned}
\left.\frac{\partial G}{\partial \theta_0}\right|_{\theta_0=\pi/2} &= \frac{\rho\rho_0\cos\theta}{4\pi}\left[\frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)\sin\theta]^{3/2}} - \frac{R^3}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0\cos(\phi-\phi_0)\sin\theta]^{3/2}}\right] \\
&\quad - \frac{\rho\rho_0\cos\theta}{4\pi}\left[\frac{1}{[\rho^2+\rho_0^2+2\rho\rho_0\cos(\phi+\phi_0)\sin\theta]^{3/2}} - \frac{R^3}{[\rho^2\rho_0^2+R^4+2R^2\rho\rho_0\cos(\phi+\phi_0)\sin\theta]^{3/2}}\right] \\
&\quad - \frac{\rho\rho_0\cos\theta}{4\pi}\left[\frac{1}{[\rho^2+\rho_0^2+2\rho\rho_0\cos(\phi+\phi_0)\sin\theta]^{3/2}} - \frac{R^3}{[\rho^2\rho_0^2+R^4+2R^2\rho\rho_0\cos(\phi+\phi_0)\sin\theta]^{3/2}}\right] \\
&\quad + \frac{\rho\rho_0\cos\theta}{4\pi}\left[\frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)\sin\theta]^{3/2}} - \frac{R^3}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0\cos(\phi-\phi_0)\sin\theta]^{3/2}}\right] \\
&\quad + \frac{\rho\rho_0\cos\theta}{4\pi}\left[\frac{1}{[\rho^2+\rho_0^2+2\rho\rho_0\cos(\phi-\phi_0)\sin\theta]^{3/2}} - \frac{R^3}{[\rho^2\rho_0^2+R^4+2R^2\rho\rho_0\cos(\phi-\phi_0)\sin\theta]^{3/2}}\right] \\
&\quad - \frac{\rho\rho_0\cos\theta}{4\pi}\left[\frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi+\phi_0)\sin\theta]^{3/2}} - \frac{R^3}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0\cos(\phi+\phi_0)\sin\theta]^{3/2}}\right] \\
&\quad - \frac{\rho\rho_0\cos\theta}{4\pi}\left[\frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi+\phi_0)\sin\theta]^{3/2}} - \frac{R^3}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0\cos(\phi+\phi_0)\sin\theta]^{3/2}}\right] \\
&\quad + \frac{\rho\rho_0\cos\theta}{4\pi}\left[\frac{1}{[\rho^2+\rho_0^2+2\rho\rho_0\cos(\phi-\phi_0)\sin\theta]^{3/2}} - \frac{R^3}{[\rho^2\rho_0^2+R^4+2R^2\rho\rho_0\cos(\phi-\phi_0)\sin\theta]^{3/2}}\right] \\
&= \frac{\rho\rho_0\cos\theta}{2\pi}\left[\frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)\sin\theta]^{3/2}} - \frac{R^3}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0\cos(\phi-\phi_0)\sin\theta]^{3/2}}\right] \\
&\quad - \frac{\rho\rho_0\cos\theta}{2\pi}\left[\frac{1}{[\rho^2+\rho_0^2+2\rho\rho_0\cos(\phi+\phi_0)\sin\theta]^{3/2}} - \frac{R^3}{[\rho^2\rho_0^2+R^4+2R^2\rho\rho_0\cos(\phi+\phi_0)\sin\theta]^{3/2}}\right] \\
&\quad + \frac{\rho\rho_0\cos\theta}{2\pi}\left[\frac{1}{[\rho^2+\rho_0^2+2\rho\rho_0\cos(\phi-\phi_0)\sin\theta]^{3/2}} - \frac{R^3}{[\rho^2\rho_0^2+R^4+2R^2\rho\rho_0\cos(\phi-\phi_0)\sin\theta]^{3/2}}\right] \\
&\quad - \frac{\rho\rho_0\cos\theta}{2\pi}\left[\frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi+\phi_0)\sin\theta]^{3/2}} - \frac{R^3}{[\rho^2\rho_0^2+R^4-2R^2\rho\rho_0\cos(\phi+\phi_0)\sin\theta]^{3/2}}\right]
\end{aligned}$$