Exercise 15

(a) Show that if $v(x, y)$ is harmonic, so is $u(x, y) = v(x^2 - y^2, 2xy)$.

(b) Show that the transformation $(x, y) \longmapsto (x^2 - y^2, 2xy)$ maps the first quadrant onto the half-plane $\{y > 0\}$. (*Hint:* Use either polar coordinates of complex variables.)

Solution

A complex variable $z = x + iy$ can be written in polar form as $z = re^{i\theta}$. If the point (x, y) lies in the first quadrant, then $0 < \theta < \pi/2$. Squaring both sides of the polar form gives $z^2 = r^2 e^{i(2\theta)}$, which means that z^2 is defined in the upper half-plane $(0 < 2\theta < \pi)$.

$$\begin{aligned} z^2 &= (x + iy)^2 \\ &= x^2 + 2ixy + i^2y^2 \\ &= x^2 + 2ixy - y^2 \\ &= (x^2 - y^2) + i(2xy) \end{aligned}$$

Suppose that $v(x, y)$ is harmonic. Then it satisfies the Laplace equation.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Make the change of variables, $r(x, y) = x^2 - y^2$ and $s(x, y) = 2xy$. Write the derivatives of v in terms of these new variables by using the chain rule.

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial v}{\partial r}(2x) + \frac{\partial v}{\partial s}(2y) = 2x\frac{\partial v}{\partial r} + 2y\frac{\partial v}{\partial s}$$

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{\partial v}{\partial r} + 2y\frac{\partial v}{\partial s}\right) = 2\frac{\partial v}{\partial r} + 2x\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial r}\right) + 2y\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial s}\right) \\ &= 2\frac{\partial v}{\partial r} + 2x\left(2x\frac{\partial}{\partial r} + 2y\frac{\partial}{\partial s}\right)\left(\frac{\partial v}{\partial r}\right) + 2y\left(2x\frac{\partial}{\partial r} + 2y\frac{\partial}{\partial s}\right)\left(\frac{\partial v}{\partial s}\right) \\ &= 2\frac{\partial v}{\partial r} + 4x^2\frac{\partial^2 v}{\partial r^2} + 4xy\frac{\partial^2 v}{\partial s \partial r} + 4yx\frac{\partial^2 v}{\partial r \partial s} + 4y^2\frac{\partial^2 v}{\partial s^2} \end{aligned}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} = \frac{\partial v}{\partial r}(-2y) + \frac{\partial v}{\partial s}(2x) = -2y\frac{\partial v}{\partial r} + 2x\frac{\partial v}{\partial s}$$

$$\begin{aligned} \frac{\partial^2 v}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(-2y\frac{\partial v}{\partial r} + 2x\frac{\partial v}{\partial s}\right) = -2\frac{\partial v}{\partial r} - 2y\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial r}\right) + 2x\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial s}\right) \\ &= -2\frac{\partial v}{\partial r} - 2y\left(-2y\frac{\partial}{\partial r} + 2x\frac{\partial}{\partial s}\right)\left(\frac{\partial v}{\partial r}\right) + 2x\left(-2y\frac{\partial}{\partial r} + 2x\frac{\partial}{\partial s}\right)\left(\frac{\partial v}{\partial s}\right) \\ &= -2\frac{\partial v}{\partial r} + 4y^2\frac{\partial^2 v}{\partial r^2} - 4yx\frac{\partial^2 v}{\partial s \partial r} - 4xy\frac{\partial^2 v}{\partial r \partial s} + 4x^2\frac{\partial^2 v}{\partial s^2} \end{aligned}$$

Add the two second derivatives together.

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} &= \left(\cancel{2\frac{\partial v}{\partial r}}+4x^2\frac{\partial^2 v}{\partial r^2}+\cancel{4xy\frac{\partial^2 v}{\partial s\partial r}}+\cancel{4yx\frac{\partial^2 v}{\partial r\partial s}}+4y^2\frac{\partial^2 v}{\partial s^2}\right)\\ &\quad+\left(\cancel{-2\frac{\partial v}{\partial r}}+4y^2\frac{\partial^2 v}{\partial r^2}-\cancel{4yx\frac{\partial^2 v}{\partial s\partial r}}-\cancel{4xy\frac{\partial^2 v}{\partial r\partial s}}+4x^2\frac{\partial^2 v}{\partial s^2}\right)\\ &=(4x^2+4y^2)\frac{\partial^2 v}{\partial r^2}+(4y^2+4x^2)\frac{\partial^2 v}{\partial s^2}\\ &=(4x^2+4y^2)\left(\frac{\partial^2 v}{\partial r^2}+\frac{\partial^2 v}{\partial s^2}\right)\end{aligned}$$

The transformed PDE that results from making the change of variables is then

$$(4x^2+4y^2)\left(\frac{\partial^2 v}{\partial r^2}+\frac{\partial^2 v}{\partial s^2}\right)=0.$$

Divide both sides by $4x^2+4y^2$ to see that this is still the Laplace equation.

$$\frac{\partial^2 v}{\partial r^2}+\frac{\partial^2 v}{\partial s^2}=0$$

Therefore, by making the change of variables, $r(x,y)=x^2-y^2$ and $s(x,y)=2xy$, the Laplace equation in the quarter plane $(x>0,\ y>0)$ can be changed to the same equation in the upper half-plane $(-\infty<x<\infty,\ y>0)$.