Exercise 16

Use Exercises 15 and 7 to find the harmonic function $u(x, y)$ in the first quadrant that has the boundary values $u(x, 0) = A$, $u(0, y) = B$, where A and B are constants. (*Hint:* $u(x, 0) = v(x^2, 0)$, etc.)

Solution

A harmonic function is a function that satisfies the Laplace equation. Here it will be solved in a quarter-plane with two prescribed boundary conditions, one along each edge of the first quadrant in the xy-plane.

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad x > 0,\ y > 0 \\ &u(x, 0) = A \\ &u(0, y) = B \end{aligned}$$

In order to turn this problem into one over the upper half-plane, make the change of variables, $r(x, y) = x^2 - y^2$ and $s(x, y) = 2xy$. According to the result in Exercise 15, the PDE transforms into the same equation but in terms of r and s.

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial s^2} = 0, \qquad -\infty < r < \infty,\ s > 0$$

Now the boundary conditions will be transformed by finding what values r and s take along the x- and y-axes.

$$\begin{aligned} y = 0: \qquad & r(x, 0) = x^2, \qquad s(x, 0) = 0 \qquad \Rightarrow \qquad u(x^2, 0) = A \\ x = 0: \qquad & r(0, y) = -y^2, \qquad s(0, y) = 0 \qquad \Rightarrow \qquad u(-y^2, 0) = B \end{aligned}$$

Since x^2 and y^2 are positive, u is A for negative values of its first argument and B for positive values of its first argument. According to Exercise 7, when solving the Laplace equation in the upper half-plane with a piecewise boundary condition like this, the solution is a function of r/s: $u(r, s) = f(r/s)$. Substituting this into the PDE results in an ODE for f.

$$(1 + z^2) f''(z) = -2z f'(z)$$

Solving it yields

$$f(z) = C_1 \tan^{-1} z + C_2,$$

which means

$$u(r, s) = C_1 \tan^{-1}\left(\frac{r}{s}\right) + C_2.$$

In the limit of u as $s \to 0$, the sign of r matters because $\tan^{-1}(\pm\infty) = \pm\pi/2$.

$$\lim_{s \to 0} u(r, s) = \begin{cases} C_1 \left(\frac{\pi}{2}\right) + C_2 & r > 0 \\ C_1 \left(-\frac{\pi}{2}\right) + C_2 & r < 0 \end{cases}$$

Apply the transformed boundary conditions to determine C_1 and C_2.

$$\begin{aligned} C_1 \left(\frac{\pi}{2}\right) + C_2 &= A \\ C_1 \left(-\frac{\pi}{2}\right) + C_2 &= B \end{aligned}$$

Solving this system of equations yields

$$C_1 = \frac{A - B}{\pi} \quad \text{and} \quad C_2 = \frac{A + B}{2}.$$

The transformed solution is then

$$u(r, s) = \frac{A - B}{\pi} \tan^{-1}\left(\frac{r}{s}\right) + \frac{A + B}{2}.$$

Therefore, changing back to x and y,

$$u(x, y) = \frac{A - B}{\pi} \tan^{-1}\left(\frac{x^2 - y^2}{2xy}\right) + \frac{A + B}{2}.$$