Exercise 17

(a) Find the Green's function for the quadrant

$$Q = \{(x, y) : \ x > 0, \ y > 0\}.$$

(*Hint:* Either use the method of reflection or reduce to the half-plane problem by the transformation in Exercise 15.)

(b) Use your answer in part (a) to solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0 \text{ in } Q, \quad u(0, y) = g(y) \text{ for } y > 0,$$
$$u(x, 0) = h(x) \text{ for } x > 0.$$

Solution

The Poisson equation will be solved in the quarter-plane with two prescribed boundary conditions on the x- and y-axes.

$$\begin{aligned} &\Delta u = f(x, y), \qquad x > 0, \ y > 0 \\ &u(0, y) = g(y) \\ &u(x, 0) = h(x) \end{aligned}$$

A Green's function representation for the solution can be obtained from Green's second identity,

$$\iint_Q (u\Delta v - v\Delta u)\, dA = \int_{\text{bdy } Q} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds.$$

Let $v = G = G(x, y; x_0, y_0)$ be the Green's function.

$$\iint_Q (u\Delta G - G\Delta u)\, dA = \int_{\text{bdy } Q} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) ds$$

If we require it to satisfy

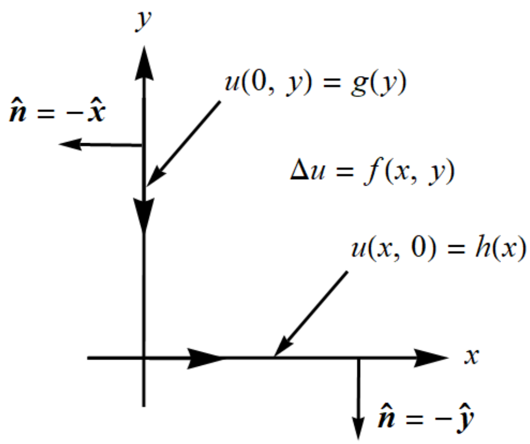
$$\begin{aligned} &\Delta G = \delta(x - x_0)\delta(y - y_0), \qquad x > 0, \ y > 0 \\ &G = 0 \quad \text{on } x = 0, \, y = 0, \end{aligned}$$

where (x_0, y_0) is a point in the first quadrant, then the identity becomes

$$\iint_Q [u(x, y)\delta(x - x_0)\delta(y - y_0) - G(x, y; x_0, y_0)f(x, y)]\, dA = \int_{\text{bdy } Q} \left(u\frac{\partial G}{\partial n} - (0)\frac{\partial u}{\partial n}\right) ds.$$

The normal derivative of G can be written as $\partial G/\partial n = \nabla G \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the outward unit vector normal to the boundary.

$$\iint_Q u(x, y)\delta(x - x_0)\delta(y - y_0)\, dA - \iint_Q G(x, y; x_0, y_0)f(x, y)\, dA = \int_{\text{bdy } Q} u\nabla G \cdot \hat{\mathbf{n}}\, ds$$



Since (x_0, y_0) lies in the first quadrant, the integral on the left involving the delta functions is $u(x_0, y_0)$. Also, integrating around the boundary counterclockwise yields two integrals, one along the y-axis and one alone the x-axis.

$$\begin{aligned} u(x_0,y_0) - \int_0^\infty \int_0^\infty G(x,y;x_0,y_0) f(x,y)\,dx\,dy &= \int_\infty^0 u(0,y)\nabla G\cdot(-\hat{\mathbf{x}})\bigg|_{x=0}(-dy) + \int_0^\infty u(x,0)\nabla G\cdot(-\hat{\mathbf{y}})\bigg|_{y=0} dx \\ &= \int_0^\infty g(y)\left(-\frac{\partial G}{\partial x}\right)\bigg|_{x=0} dy + \int_0^\infty h(x)\left(-\frac{\partial G}{\partial y}\right)\bigg|_{y=0} dx \\ &= -\int_0^\infty g(y)\frac{\partial G}{\partial x}\bigg|_{x=0} dy - \int_0^\infty h(x)\frac{\partial G}{\partial y}\bigg|_{y=0} dx \end{aligned}$$

Solve this equation for u.

$$u(x_0,y_0) = \int_0^\infty \int_0^\infty G(x,y;x_0,y_0) f(x,y)\,dx\,dy - \int_0^\infty g(y)\frac{\partial G}{\partial x}\bigg|_{x=0} dy - \int_0^\infty h(x)\frac{\partial G}{\partial y}\bigg|_{y=0} dx$$

Switch the roles of x_0 and y_0 with those of x and y, respectively.

$$u(x,y) = \int_0^\infty \int_0^\infty G(x_0,y_0;x,y) f(x_0,y_0)\,dx_0\,dy_0 - \int_0^\infty g(y_0)\frac{\partial G}{\partial x_0}\bigg|_{x_0=0} dy_0 - \int_0^\infty h(x_0)\frac{\partial G}{\partial y_0}\bigg|_{y_0=0} dx_0$$

Therefore, using the fact that the Green's function is symmetric,

$$u(x,y) = \int_0^\infty \int_0^\infty G(x,y;x_0,y_0) f(x_0,y_0)\,dx_0\,dy_0 - \int_0^\infty g(y_0)\frac{\partial G}{\partial x_0}\bigg|_{x_0=0} dy_0 - \int_0^\infty h(x_0)\frac{\partial G}{\partial y_0}\bigg|_{y_0=0} dx_0.$$

The solution for Poisson's equation is known, then, if the Green's function in the quarter-plane can be determined. Begin by finding the Green's function in the whole plane (no boundaries).

$$\Delta g = \delta(x-x_0)\delta(y-y_0), \qquad -\infty < x < \infty,\ -\infty < y < \infty$$

g can be interpreted as the electrostatic potential, and $\delta(x-x_0)\delta(y-y_0)$ can be interpreted as the charge density for a unit charge located at (x_0, y_0). Since there are no boundaries, g is expected to vary solely as a function of the radial distance from (x_0, y_0): $g = g(r)$, where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$. Integrate both sides over a disk centered at (x_0, y_0) with radius r.

$$\iint\limits_{(x-x_0)^2+(y-y_0)^2 \le r^2} \Delta g\,dA = \iint\limits_{(x-x_0)^2+(y-y_0)^2 \le r^2} \delta(x-x_0)\delta(y-y_0)\,dA$$

Since the disk contains (x_0, y_0), the right side is 1. Write the Laplacian operator as $\Delta = \nabla^2$

$$\iint\limits_{(x-x_0)^2+(y-y_0)^2\leq \mathscr{r}^2} \nabla^2 g\, dA = 1$$

$$\iint\limits_{(x-x_0)^2+(y-y_0)^2\leq \mathscr{r}^2} \nabla\cdot\nabla g\, dA = 1$$

and apply the two-dimensional divergence theorem.

$$\int\limits_{(x-x_0)^2+(y-y_0)^2= \mathscr{r}^2} \nabla g\cdot\hat{\boldsymbol{\mathscr{r}}}\, ds = 1$$

Here $\hat{\boldsymbol{\mathscr{r}}}$ is the unit vector normal to this disk at every point on the boundary.

$$\int\limits_{(x-x_0)^2+(y-y_0)^2= \mathscr{r}^2} \frac{dg}{d\mathscr{r}}\, ds = 1$$

Because g only depends on $\mathscr{r}$, its derivative is constant on the disk's boundary.

$$\frac{dg}{d\mathscr{r}}\int\limits_{(x-x_0)^2+(y-y_0)^2= \mathscr{r}^2} ds = 1$$

This line integral is just the disk's circumference.

$$\frac{dg}{d\mathscr{r}}(2\pi\mathscr{r}) = 1$$

Divide both sides by $2\pi\mathscr{r}$.

$$\frac{dg}{d\mathscr{r}} = \frac{1}{2\pi\mathscr{r}}$$

Integrate both sides with respect to $\mathscr{r}$.

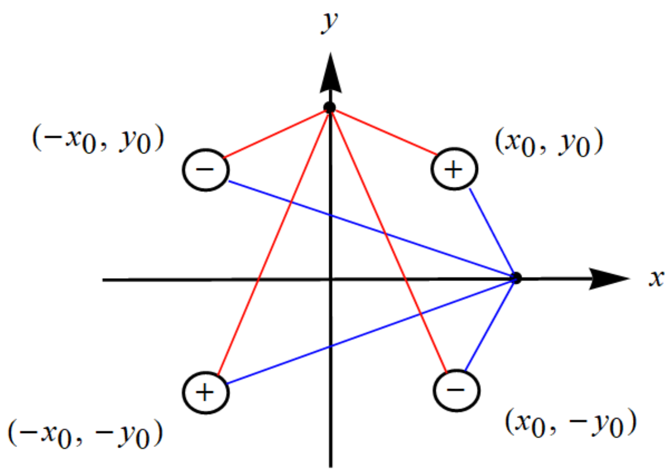
$$g(\mathscr{r}) = \frac{1}{2\pi}\ln\mathscr{r}$$

The Green's function for the whole plane is then

$$g(x, y; x_0, y_0) = \frac{1}{2\pi}\ln\sqrt{(x-x_0)^2+(y-y_0)^2}.$$

Now that it's known, the Green's function for the upper half-plane can be determined by the method of images. A convocation of point charges in the whole plane will be arranged so that the boundary condition, $G = 0$ on $x = 0$ and $y = 0$, is satisfied.

For a positive unit charge located at (x_0, y_0), two negative unit charges at $(-x_0, y_0)$ and $(x_0, -y_0)$ and one positive unit charge at $(-x_0, -y_0)$ should be placed. This way, the potential due to each positive charge is cancelled by that due to a negative charge at every point on the boundary.



The quarter-plane Green's function can now be written.

$$G(x, y; x_0, y_0) = +g(x, y; x_0, y_0) - g(x, y; -x_0, y_0) - g(x, y; x_0, -y_0) + g(x, y; -x_0, -y_0), \quad x > 0,\ y > 0$$

Since g is defined over the whole plane, it's important to note the restriction to $x > 0$, $y > 0$ for G.

$$\begin{aligned}
G(x, y; x_0, y_0) &= \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2} - \frac{1}{2\pi} \ln \sqrt{(x + x_0)^2 + (y - y_0)^2} \\
&\qquad - \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y + y_0)^2} + \frac{1}{2\pi} \ln \sqrt{(x + x_0)^2 + (y + y_0)^2} \\
&= \frac{1}{2\pi} \ln \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{\sqrt{(x + x_0)^2 + (y - y_0)^2}} + \frac{1}{2\pi} \ln \frac{\sqrt{(x + x_0)^2 + (y + y_0)^2}}{\sqrt{(x - x_0)^2 + (y + y_0)^2}} \\
&= \frac{1}{2\pi} \ln \left[\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{\sqrt{(x + x_0)^2 + (y - y_0)^2}} \frac{\sqrt{(x + x_0)^2 + (y + y_0)^2}}{\sqrt{(x - x_0)^2 + (y + y_0)^2}} \right] \\
&= \frac{1}{4\pi} \ln \frac{[(x - x_0)^2 + (y - y_0)^2][(x + x_0)^2 + (y + y_0)^2]}{[(x + x_0)^2 + (y - y_0)^2][(x - x_0)^2 + (y + y_0)^2]}
\end{aligned}$$

Calculate $\partial G/\partial x_0$

$$\frac{\partial G}{\partial x_0} = \frac{1}{4\pi} \frac{-2(x - x_0)}{(x - x_0)^2 + (y - y_0)^2} - \frac{1}{4\pi} \frac{2(x + x_0)}{(x + x_0)^2 + (y - y_0)^2} - \frac{1}{4\pi} \frac{-2(x - x_0)}{(x - x_0)^2 + (y + y_0)^2} + \frac{1}{4\pi} \frac{2(x + x_0)}{(x + x_0)^2 + (y + y_0)^2}$$

and evaluate it at $x_0 = 0$.

$$\begin{aligned}
\left.\frac{\partial G}{\partial x_0}\right|_{x_0 = 0} &= \frac{1}{4\pi} \frac{-2x}{x^2 + (y - y_0)^2} - \frac{1}{4\pi} \frac{2x}{x^2 + (y - y_0)^2} - \frac{1}{4\pi} \frac{-2x}{x^2 + (y + y_0)^2} + \frac{1}{4\pi} \frac{2x}{x^2 + (y + y_0)^2} \\
&= -\frac{x}{\pi} \left[\frac{1}{x^2 + (y - y_0)^2} - \frac{1}{x^2 + (y + y_0)^2} \right]
\end{aligned}$$

Calculate $\partial G/\partial y_0$

$$\frac{\partial G}{\partial y_0} = \frac{1}{4\pi}\frac{-2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{1}{4\pi}\frac{-2(y-y_0)}{(x+x_0)^2+(y-y_0)^2} - \frac{1}{4\pi}\frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2} + \frac{1}{4\pi}\frac{2(y+y_0)}{(x+x_0)^2+(y+y_0)^2}$$

and evaluate it at $y_0 = 0$.

$$\begin{aligned}\left.\frac{\partial G}{\partial y_0}\right|_{y_0=0} &= \frac{1}{4\pi}\frac{-2y}{(x-x_0)^2+y^2} - \frac{1}{4\pi}\frac{-2y}{(x+x_0)^2+y^2} - \frac{1}{4\pi}\frac{2y}{(x-x_0)^2+y^2} + \frac{1}{4\pi}\frac{2y}{(x+x_0)^2+y^2} \\ &= -\frac{y}{\pi}\left[\frac{1}{(x-x_0)^2+y^2} - \frac{1}{(x+x_0)^2+y^2}\right]\end{aligned}$$

Therefore,

$$\begin{aligned}u(x,y) = \frac{1}{4\pi}\int_0^\infty\int_0^\infty f(x_0,y_0)\ln\frac{[(x-x_0)^2+(y-y_0)^2][(x+x_0)^2+(y+y_0)^2]}{[(x+x_0)^2+(y-y_0)^2][(x-x_0)^2+(y+y_0)^2]}\,dx_0\,dy_0 \\ + \frac{x}{\pi}\int_0^\infty g(y_0)\left[\frac{1}{x^2+(y-y_0)^2} - \frac{1}{x^2+(y+y_0)^2}\right]dy_0 \\ + \frac{y}{\pi}\int_0^\infty h(x_0)\left[\frac{1}{(x-x_0)^2+y^2} - \frac{1}{(x+x_0)^2+y^2}\right]dx_0.\end{aligned}$$