Exercise 19

Consider the four-dimensional laplacian $\Delta u = u_{xx} + u_{yy} + u_{zz} + u_{ww}$. Show that its fundamental solution is r^{-2}, where $r^2 = x^2 + y^2 + z^2 + w^2$.

Solution

Consider the Poisson equation over infinite four-dimensional space D.

$$\Delta u = f(x, y, z, w), \qquad -\infty < x, y, z, w < \infty$$

A Green's function representation for the solution can be obtained from Green's second identity,

$$\iiiint\limits_D (u\Delta v - v\Delta u)\, dV = \iiint\limits_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS.$$

Let $v = G = G(x, y, z, w; x_0, y_0, z_0, w_0)$ be the Green's function. Since there are no boundaries, the right side is zero.

$$\iiiint\limits_D (u\Delta G - G\Delta u)\, dV = 0$$

If we require the Green's function to satisfy

$$\Delta G = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(w - w_0), \qquad -\infty < x, y, z, w < \infty,$$

where (x_0, y_0, z_0, w_0) is a point in hyperspace, then

$$\iiiint\limits_D [u(x, y, z, w)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(w - w_0) - G(x, y, z, w; x_0, y_0, z_0, w_0)f(x, y, z, w)]\, dV = 0.$$

Split up the integral into two.

$$\iiiint\limits_D u(x, y, z, w)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(w - w_0)\, dV - \iiiint\limits_D G(x, y, z, w; x_0, y_0, z_0, w_0)f(x, y, z, w)\, dV = 0$$

The one involving the delta functions is $u(x_0, y_0, z_0, w_0)$.

$$u(x_0, y_0, z_0, w_0) - \iiiint\limits_D G(x, y, z, w; x_0, y_0, z_0, w_0)f(x, y, z, w)\, dV = 0$$

Solve for u.

$$u(x_0, y_0, z_0, w_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y, z, w; x_0, y_0, z_0, w_0)f(x, y, z, w)\, dx\, dy\, dz\, dw$$

Switch the roles of x, y, z, and w with those of x_0, y_0, z_0, and w_0, respectively.

$$u(x, y, z, w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x_0, y_0, z_0, w_0; x, y, z, w)f(x_0, y_0, z_0, w_0)\, dx_0\, dy_0\, dz_0\, dw_0$$

Use the fact that the Green's function is symmetric.

$$u(x,y,z,w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,z,w;x_0,y_0,z_0,w_0) f(x_0,y_0,z_0,w_0)\,dx_0\,dy_0\,dz_0\,dw_0$$

The fundamental solution of the Poisson equation is obtained by setting $f(x,y,z,w) = \delta(x)\delta(y)\delta(z)\delta(w)$. As a result,

$$\begin{aligned} u(x,y,z,w) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,z,w;x_0,y_0,z_0,w_0)\delta(x_0)\delta(y_0)\delta(z_0)\delta(w_0)\,dx_0\,dy_0\,dz_0\,dw_0 \\ &= G(x,y,z,w;0,0,0,0). \end{aligned}$$

The aim now is to solve the defining PDE for $G(x,y,z,w;0,0,0,0)$.

$$\Delta G = \delta(x)\delta(y)\delta(z)\delta(w), \qquad -\infty < x,y,z,w < \infty$$

G can be interpreted as the electrostatic potential, and $\delta(x)\delta(y)\delta(z)\delta(w)$ can be interpreted as the charge density for a unit charge located at the origin $(0,0,0,0)$. Since there are no boundaries, G is expected to vary solely as a function of the radial distance from $(0,0,0,0)$: $G = G(r)$, where $r = \sqrt{x^2+y^2+z^2+w^2}$. Away from the origin, the PDE is homogeneous.

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} + \frac{\partial^2 G}{\partial w^2} = 0, \qquad (x,y,z,w) \neq (0,0,0,0)$$

Make the change of variables $\xi = x^2+y^2+z^2+w^2$ and use the chain rule to write each of the derivatives in terms of this new variable.

$$\frac{\partial G}{\partial x} = \frac{dG}{d\xi}\frac{\partial \xi}{\partial x} = \frac{dG}{d\xi}(2x)$$

$$\frac{\partial^2 G}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{dG}{d\xi}\right) = 2\frac{dG}{d\xi} + 2x\frac{\partial}{\partial x}\left(\frac{dG}{d\xi}\right) = 2\frac{dG}{d\xi} + 2x\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}\left(\frac{dG}{d\xi}\right) = 2\frac{dG}{d\xi} + 4x^2\frac{d^2G}{d\xi^2}$$

$$\frac{\partial G}{\partial y} = \frac{dG}{d\xi}\frac{\partial \xi}{\partial y} = \frac{dG}{d\xi}(2y)$$

$$\frac{\partial^2 G}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial y}\right) = \frac{\partial}{\partial y}\left(2y\frac{dG}{d\xi}\right) = 2\frac{dG}{d\xi} + 2y\frac{\partial}{\partial y}\left(\frac{dG}{d\xi}\right) = 2\frac{dG}{d\xi} + 2y\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi}\left(\frac{dG}{d\xi}\right) = 2\frac{dG}{d\xi} + 4y^2\frac{d^2G}{d\xi^2}$$

$$\frac{\partial G}{\partial z} = \frac{dG}{d\xi}\frac{\partial \xi}{\partial z} = \frac{dG}{d\xi}(2z)$$

$$\frac{\partial^2 G}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial G}{\partial z}\right) = \frac{\partial}{\partial z}\left(2z\frac{dG}{d\xi}\right) = 2\frac{dG}{d\xi} + 2z\frac{\partial}{\partial z}\left(\frac{dG}{d\xi}\right) = 2\frac{dG}{d\xi} + 2z\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}\left(\frac{dG}{d\xi}\right) = 2\frac{dG}{d\xi} + 4z^2\frac{d^2G}{d\xi^2}$$

$$\frac{\partial G}{\partial w} = \frac{dG}{d\xi}\frac{\partial \xi}{\partial w} = \frac{dG}{d\xi}(2w)$$

$$\frac{\partial^2 G}{\partial w^2} = \frac{\partial}{\partial w}\left(\frac{\partial G}{\partial w}\right) = \frac{\partial}{\partial w}\left(2w\frac{dG}{d\xi}\right) = 2\frac{dG}{d\xi} + 2w\frac{\partial}{\partial w}\left(\frac{dG}{d\xi}\right) = 2\frac{dG}{d\xi} + 2w\frac{\partial \xi}{\partial w}\frac{\partial}{\partial \xi}\left(\frac{dG}{d\xi}\right) = 2\frac{dG}{d\xi} + 4w^2\frac{d^2G}{d\xi^2}$$

The transformed PDE is

$$8\frac{dG}{d\xi} + 4(x^2+y^2+z^2+w^2)\frac{d^2G}{d\xi^2} = 0,$$

or

$$8\frac{dG}{d\xi} + 4\xi\frac{d^2G}{d\xi^2} = 0.$$

Multiply both sides by ξ.

$$8\xi\frac{dG}{d\xi} + 4\xi^2\frac{d^2G}{d\xi^2} = 0$$

The left side can be written as a derivative by the product rule.

$$\frac{d}{d\xi}\left(4\xi^2\frac{dG}{d\xi}\right) = 0$$

Integrate both sides with respect to ξ.

$$4\xi^2\frac{dG}{d\xi} = C_1$$

Divide both sides by $4\xi^2$.

$$\frac{dG}{d\xi} = \frac{C_1}{4\xi^2}$$

Integrate both sides with respect to ξ once more.

$$G(\xi) = -\frac{C_1}{4\xi} + C_2$$

If we require G to be zero as $\xi \to \infty$, then $C_2 = 0$.

$$G(\xi) = -\frac{C_1}{4\xi}$$

Because the PDE is homogeneous away from the origin, C_1 can be anything. Here it's set to -4 so that there is no constant factor.

$$G(\xi) = \frac{1}{\xi}$$

In terms of the original variables, this is

$$G(x, y, z, w; 0, 0, 0, 0) = \frac{1}{x^2 + y^2 + z^2 + w^2}.$$

Therefore, the fundamental solution to the four-dimensional Poisson equation is

$$u(x, y, z, w) = \frac{1}{x^2 + y^2 + z^2 + w^2}.$$