Exercise 21

The *Neumann function* $N(x, y)$ for a domain D is defined exactly like the Green's function in Section 7.3 except that (ii) is replaced by the Neumann boundary condition

$$\frac{\partial N}{\partial n} = c \quad \text{for } x \in \text{bdy } D$$

for a suitable constant c.

(a) Show that $c = 1/A$, where A is the area of bdy D. ($c = 0$ if $A = \infty$)

(b) State and prove the analog of Theorem 7.3.1, expressing the solution of the Neumann problem in terms of the Neumann function.

Solution

Consider the Poisson equation in some domain D that is subject to a Neumann condition on the boundary of D.

$$\begin{aligned} \Delta U &= f(x, y, z) \quad \text{in } D \\ \frac{\partial U}{\partial n} &= g(x, y, z) \quad \text{on bdy } D \end{aligned}$$

There is also a solvability condition for the boundary value problem, which is obtained by integrating both sides of the PDE over the volume of D.

$$\iiint_D \Delta U \, dV = \iiint_D f(x, y, z) \, dV$$

$$\iiint_D \nabla^2 U \, dV = \iiint_D f(x, y, z) \, dV$$

$$\iiint_D \nabla \cdot \nabla U \, dV = \iiint_D f(x, y, z) \, dV$$

Apply the divergence theorem on the left side. Let $\hat{\mathbf{n}}$ be an outward unit vector normal to the boundary.

$$\iint_{\text{bdy } D} \nabla U \cdot \hat{\mathbf{n}} \, dS = \iiint_D f(x, y, z) \, dV$$

$$\iint_{\text{bdy } D} \frac{\partial U}{\partial n} \, dS = \iiint_D f(x, y, z) \, dV$$

$$\iint_{\text{bdy } D} g(x, y, z) \, dS = \iiint_D f(x, y, z) \, dV$$

What this means is that the prescribed functions, f and g, are not arbitrary and must satisfy this relationship for there to be a solution to the problem.

Part (a)

Green's second identity holds for any two functions, u and v, on a domain D and its boundary.

$$\iiint_D (u\Delta v - v\Delta u)\, dV = \iint_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$$

Let $u = 1$ and let $v = N = N(x, y, z; x_0, y_0, z_0)$ be the Neumann function.

$$\iiint_D \Delta N\, dV = \iint_{\text{bdy } D} \frac{\partial N}{\partial n}\, dS$$

If we require the Neumann function to satisfy

$$\begin{aligned} &\Delta N = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \quad \text{in } D \\ &\frac{\partial N}{\partial n} = c \quad \text{on bdy } D, \end{aligned}$$

where (x_0, y_0, z_0) is a point in D, then the identity reduces to

$$\iiint_D \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\, dV = \iint_{\text{bdy } D} c\, dS.$$

The volume integral of the delta functions is 1, and the constant c can be brought in front of the surface integral.

$$1 = c \iint_{\text{bdy } D} dS$$

Therefore,

$$c = \frac{1}{\displaystyle\iint_{\text{bdy } D} dS} = \frac{1}{A}.$$

Part (b)

A Neumann function representation for the solution can be obtained from Green's second identity,

$$\iiint_D (u\Delta v - v\Delta u)\,dV = \iint_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS.$$

Let $u = U(x, y, z)$ and let $v = N = N(x, y, z; x_0, y_0, z_0)$ be the Neumann function.

$$\iiint_D (U\Delta N - N\Delta U)\,dV = \iint_{\text{bdy } D} \left(U\frac{\partial N}{\partial n} - N\frac{\partial U}{\partial n}\right) dS$$

If we require N to satisfy

$$\begin{aligned} \Delta N &= \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \quad \text{in } D \\ \frac{\partial N}{\partial n} &= c \quad \text{on bdy } D, \end{aligned}$$

where (x_0, y_0, z_0) is a point in D, then the identity becomes

$$\iiint_D [U(x, y, z)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) - N(x, y, z; x_0, y_0, z_0)f(x, y, z)]\,dV$$
$$= \iint_{\text{bdy } D} [U(x, y, z)c - N(x, y, z; x_0, y_0, z_0)g(x, y, z)]\,dS.$$

Split up the integrals.

$$\iiint_D U(x, y, z)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\,dV - \iiint_D N(x, y, z; x_0, y_0, z_0)f(x, y, z)\,dV$$
$$= c\iint_{\text{bdy } D} U(x, y, z)\,dS - \iint_{\text{bdy } D} N(x, y, z; x_0, y_0, z_0)g(x, y, z)\,dS$$

The volume integral involving the delta functions is $U(x_0, y_0, z_0)$. Rewrite c using the formula found for it in part (a).

$$U(x_0, y_0, z_0) - \iiint_D N(x, y, z; x_0, y_0, z_0)f(x, y, z)\,dV$$
$$= \frac{\displaystyle\iint_{\text{bdy } D} U(x, y, z)\,dS}{\displaystyle\iint_{\text{bdy } D} dS} - \iint_{\text{bdy } D} N(x, y, z; x_0, y_0, z_0)g(x, y, z)\,dS$$

The first term on the right side is the average value of U over the boundary of D, a constant. Denote it as $\overline{U}$.

$$U(x_0, y_0, z_0) - \iiint_D N(x, y, z; x_0, y_0, z_0)f(x, y, z)\,dV = \overline{U} - \iint_{\text{bdy } D} N(x, y, z; x_0, y_0, z_0)g(x, y, z)\,dS$$

Solve for U.

$$U(x_0, y_0, z_0) = \overline{U} + \iiint_D N(x, y, z; x_0, y_0, z_0) f(x, y, z)\, dV - \iint_{\text{bdy } D} N(x, y, z; x_0, y_0, z_0) g(x, y, z)\, dS$$

Switch the roles of x_0, y_0, and z_0 with those of x, y, and z, respectively.

$$U(x, y, z) = \overline{U} + \iiint_D N(x_0, y_0, z_0; x, y, z) f(x_0, y_0, z_0)\, dV_0 - \iint_{\text{bdy } D} N(x_0, y_0, z_0; x, y, z) g(x_0, y_0, z_0)\, dS_0$$