Exercise 24

Solve the Neumann problem in the half-space $\{z > 0\}$.

Solution

The Poisson equation will be solved in the upper half-space D with a Neumann boundary condition.

$$\Delta U = f(x,y,z), \quad -\infty < x, y < \infty,\ z > 0$$
$$\frac{\partial U}{\partial z}(x,y,0) = h(x,y)$$

A Neumann function representation for the solution can be obtained from Green's second identity,

$$\iiint_D (u\Delta v - v\Delta u)\, dV = \iint_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS.$$

Let $u = U(x,y,z)$ and let $v = N = N(x,y,z;x_0,y_0,z_0)$ be the Neumann function.

$$\iiint_D (U\Delta N - N\Delta U)\, dV = \iint_{\text{bdy } D} \left(U\frac{\partial N}{\partial n} - N\frac{\partial U}{\partial n}\right) dS$$

If we require N to satisfy

$$\Delta N = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \quad -\infty < x, y < \infty,\ z > 0$$
$$\frac{\partial N}{\partial n} = c \quad \text{on bdy } D,$$

where c is a constant and (x_0, y_0, z_0) is a point in the upper half-space, then the identity becomes

$$\iiint_D [U(x,y,z)\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) - N(x,y,z;x_0,y_0,z_0)f(x,y,z)]\, dV$$
$$= \iint_{\text{bdy } D} \left[U(x,y,z)c - N(x,y,z;x_0,y_0,z_0)\frac{\partial U}{\partial n}\right] dS.$$

Write the normal derivative as $\partial U/\partial n = \nabla U \cdot \hat{\mathbf{n}}$ and split up the integrals.

$$\iiint_D U(x,y,z)\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\, dV - \iiint_D N(x,y,z;x_0,y_0,z_0)f(x,y,z)\, dV$$
$$= c\iint_{\text{bdy } D} U(x,y,z)\, dS - \iint_{\text{bdy } D} N(x,y,z;x_0,y_0,z_0)\nabla U \cdot \hat{\mathbf{n}}\, dS$$

The integral involving the delta functions is $U(x_0, y_0, z_0)$.

$$U(x_0,y_0,z_0) - \iiint_D N(x,y,z;x_0,y_0,z_0)f(x,y,z)\, dV$$
$$= c\iint_{\text{bdy } D} U(x,y,z)\, dS - \iint_{\text{bdy } D} N(x,y,z;x_0,y_0,z_0)\nabla U \cdot \hat{\mathbf{n}}\, dS \quad (1)$$

Determine the constant c by setting $u = 1$ and $v = N(x, y, z; x_0, y_0, z_0)$ in Green's second identity.

$$\iiint_D \Delta N \, dV = \iint_{\text{bdy } D} \frac{\partial N}{\partial n} \, dS$$

$$\iiint_D \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \, dV = \iint_{\text{bdy } D} c \, dS$$

$$1 = c \iint_{\text{bdy } D} dS$$

Solve for c.

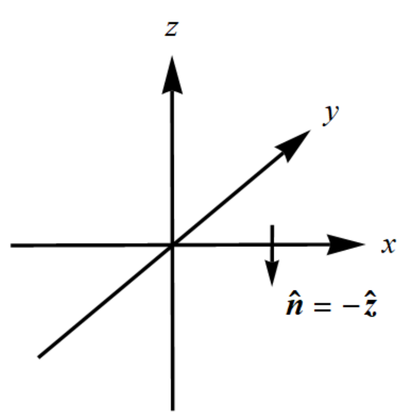
$$c = \frac{1}{\displaystyle\iint_{\text{bdy } D} dS} = \frac{1}{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx \, dy} = 0$$

Then equation (1) becomes

$$U(x_0, y_0, z_0) - \iiint_D N(x, y, z; x_0, y_0, z_0) f(x, y, z) \, dV$$
$$= \frac{\displaystyle\int_{\text{bdy } D} U(x, y, z) \, dS}{\displaystyle\int_{\text{bdy } D} dS} - \iint_{\text{bdy } D} N(x, y, z; x_0, y_0, z_0) \nabla U \cdot \hat{\mathbf{n}} \, dS.$$

The first term on the right side is the average value of U on the boundary of the upper half-space ($z = 0$), a constant. Denote it as $\overline{U}$.

$$U(x_0, y_0, z_0) - \iiint_D N(x, y, z; x_0, y_0, z_0) f(x, y, z) \, dV = \overline{U} - \iint_{\text{bdy } D} N(x, y, z; x_0, y_0, z_0) \nabla U \cdot \hat{\mathbf{n}} \, dS$$



As the figure illustrates, the outward unit vector normal to the upper half-space is $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$.

$$U(x_0, y_0, z_0) - \iiint_D N(x, y, z; x_0, y_0, z_0) f(x, y, z) \, dV = \overline{U} - \iint_{\text{bdy } D} N(x, y, z; x_0, y_0, z_0) \nabla U \cdot (-\hat{\mathbf{z}}) \, dS$$

Evaluate the dot product.

$$U(x_0,y_0,z_0) - \iiint_D N(x,y,z;x_0,y_0,z_0)f(x,y,z)\,dV = \overline{U} - \iint_{\text{bdy } D} N(x,y,z;x_0,y_0,z_0)\left(-\frac{\partial U}{\partial z}\right) dS$$

Substitute the prescribed boundary condition and write the integration limits.

$$\begin{aligned}U(x_0,y_0,z_0) - \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty N(x,y,z;x_0,y_0,z_0)f(x,y,z)\,dx\,dy\,dz \\ = \overline{U} + \int_{-\infty}^\infty \int_{-\infty}^\infty N(x,y,z;x_0,y_0,z_0)\big|_{z=0}\,h(x,y)\,dx\,dy\end{aligned}$$

Solve for U.

$$\begin{aligned}U(x_0,y_0,z_0) = \overline{U} + \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty N(x,y,z;x_0,y_0,z_0)f(x,y,z)\,dx\,dy\,dz \\ + \int_{-\infty}^\infty \int_{-\infty}^\infty N(x,y,z;x_0,y_0,z_0)\big|_{z=0}\,h(x,y)\,dx\,dy\end{aligned}$$

Switch the roles of x, y, and z with those of x_0, y_0, and z_0, respectively.

$$\begin{aligned}U(x,y,z) = \overline{U} + \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty N(x_0,y_0,z_0;x,y,z)f(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0 \\ + \int_{-\infty}^\infty \int_{-\infty}^\infty N(x_0,y_0,z_0;x,y,z)\big|_{z_0=0}\,h(x_0,y_0)\,dx_0\,dy_0 \quad (2)\end{aligned}$$

The Neumann function will now be shown to be symmetric if $c = 0$. Set $u = N(x,y,z;x_1,y_1,z_1)$ and $v = N(x,y,z;x_2,y_2,z_2)$ in Green's second identity,

$$\begin{aligned}&\iiint_D [N(x,y,z;x_1,y_1,z_1)\Delta N(x,y,z;x_2,y_2,z_2) - N(x,y,z;x_2,y_2,z_2)\Delta N(x,y,z;x_1,y_1,z_1)]\,dV \\ &= \iint_{\text{bdy } D}\left[N(x,y,z;x_1,y_1,z_1)\frac{\partial N}{\partial n}(x,y,z;x_2,y_2,z_2) - N(x,y,z;x_2,y_2,z_2)\frac{\partial N}{\partial n}(x,y,z;x_1,y_1,z_1)\right] dS,\end{aligned}$$

where (x_1,y_1,z_1) and (x_2,y_2,z_2) are points in D, and $N(x,y,z;x_1,y_1,z_1)$ and $N(x,y,z;x_2,y_2,z_2)$ satisfy

$$\begin{aligned}&\Delta N = \delta(x-x_1)\delta(y-y_1)\delta(z-z_1) \quad \text{in } D & &\Delta N = \delta(x-x_2)\delta(y-y_2)\delta(z-z_2) \quad \text{in } D \\ &\frac{\partial N}{\partial n}(x,y,z;x_1,y_1,z_1) = c \quad \text{on bdy } D & &\frac{\partial N}{\partial n}(x,y,z;x_2,y_2,z_2) = c \quad \text{on bdy } D.\end{aligned}$$

Substitute these results into the identity.

$$\begin{aligned}\iiint_D [N(x,y,z;x_1,y_1,z_1)\delta(x-x_2)\delta(y-y_2)\delta(z-z_2) - N(x,y,z;x_2,y_2,z_2)\delta(x-x_1)\delta(y-y_1)\delta(z-z_1)]\,dV \\ = \iint_{\text{bdy } D} [N(x,y,z;x_1,y_1,z_1)c - N(x,y,z;x_2,y_2,z_2)c]\,dS\end{aligned}$$

Split up the integrals on the left and bring c in front of the integral on the right.

$$\iiint_D N(x,y,z;x_1,y_1,z_1)\delta(x-x_2)\delta(y-y_2)\delta(z-z_2)\,dV - \iiint_D N(x,y,z;x_2,y_2,z_2)\delta(x-x_1)\delta(y-y_1)\delta(z-z_1)\,dV$$
$$= c\iint_{\text{bdy } D} [N(x,y,z;x_1,y_1,z_1) - N(x,y,z;x_2,y_2,z_2)]\,dS$$

If $c = 0$, then the right side is zero. Evaluate the integrals on the left.

$$N(x_2,y_2,z_2;x_1,y_1,z_1) - N(x_1,y_1,z_1;x_2,y_2,z_2) = 0$$

Therefore, $N(x_2,y_2,z_2;x_1,y_1,z_1) = N(x_1,y_1,z_1;x_2,y_2,z_2)$, and equation (2) becomes

$$U(x,y,z) = \bar{U} + \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty N(x,y,z;x_0,y_0,z_0) f(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0$$
$$+ \int_{-\infty}^\infty \int_{-\infty}^\infty N(x,y,z;x_0,y_0,z_0)\big|_{z_0=0}\, h(x_0,y_0)\,dx_0\,dy_0.$$

The solution for Poisson's equation is known, then, if the Neumann function in the upper half-plane can be determined. Begin by finding the Neumann function in the whole plane (no boundaries).

$$\Delta\mathscr{N} = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \qquad -\infty < x,y < \infty,\ z > 0$$

$\mathscr{N}$ can be interpreted as the electrostatic potential, and $\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$ can be interpreted as the charge density for a unit charge located at (x_0,y_0,z_0). Since there are no boundaries, $\mathscr{N}$ is expected to vary solely as a function of the radial distance from (x_0,y_0,z_0): $\mathscr{N} = \mathscr{N}(\imath)$, where $\imath = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$. Integrate both sides over a solid ball centered at (x_0,y_0,z_0) with radius $\imath$.

$$\iiint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le \imath^2}} \Delta\mathscr{N}\,dV = \iiint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le \imath^2}} \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\,dV$$

Since the ball contains (x_0,y_0,z_0), the right side is 1. Write the Laplacian operator Δ as ∇^2

$$\iiint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le \imath^2}} \nabla^2\mathscr{N}\,dV = 1$$

$$\iiint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le \imath^2}} \nabla\cdot\nabla\mathscr{N}\,dV = 1$$

and apply the divergence theorem.

$$\iint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2= \imath^2}} \nabla\mathscr{N}\cdot\hat{\boldsymbol{\imath}}\,dS = 1$$

Here $\hat{\boldsymbol{\imath}}$ is the unit vector normal to this ball at every point on the boundary.

$$\iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=\imath^2}} \frac{d\mathscr{N}}{d\imath}\,dS = 1$$

Because $\mathscr{N}$ only depends on $\imath$, its derivative is constant on the ball's boundary.

$$\frac{d\mathscr{N}}{d\imath}\iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=\imath^2}} dS = 1$$

This surface integral is just the ball's surface area.

$$\frac{d\mathscr{N}}{d\imath}(4\pi\imath^2) = 1$$

Divide both sides by $4\pi\imath^2$.

$$\frac{d\mathscr{N}}{d\imath} = \frac{1}{4\pi\imath^2}$$

Integrate both sides with respect to $\imath$.

$$\mathscr{N}(\imath) = -\frac{1}{4\pi\imath}$$

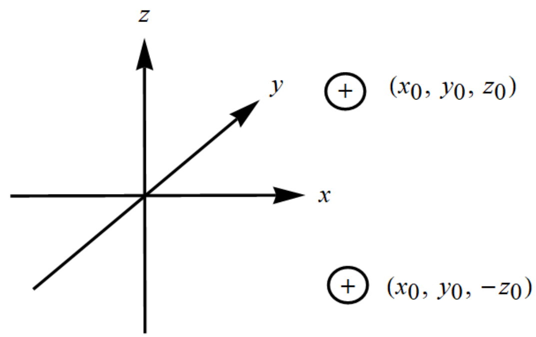
The infinite-space Neumann function is then

$$\mathscr{N}(x,y,z;x_0,y_0,z_0) = -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}.$$

Now that it's known, the Neumann function for the upper half-space can be determined by the method of images. A convocation of point charges in the whole plane will be arranged so that the boundary condition,

$$\frac{\partial N}{\partial n} = c = \frac{1}{\displaystyle\int\limits_{\text{bdy } D} ds} = \frac{1}{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy} = 0 \text{ on bdy } D \quad \Rightarrow \quad \frac{\partial N}{\partial z}(x,y,0) = 0,$$

is satisfied. This derivative of potential can be interpreted as the z-component of the electric field. For a positive unit charge at (x_0, y_0, z_0), place another positive unit charge at $(x_0, y_0, -z_0)$ so that every point on the xy-plane is equally spaced from both.



The upper half-space Neumann function can now be written.

$$N(x,y,z;x_0,y_0,z_0) = +\mathscr{N}(x,y,z;x_0,y_0,z_0) + \mathscr{N}(x,y,z;x_0,y_0,-z_0), \quad z>0$$

Because $\mathscr{N}$ is defined over infinite space, it's important to note the restriction to $z>0$ for N.

$$\begin{aligned} N(x,y,z;x_0,y_0,z_0) &= -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} - \frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}} \\ &= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right] \end{aligned}$$

Now set $z_0 = 0$ in this result.

$$\begin{aligned} N(x,y,z;x_0,y_0,z_0)\big|_{z_0=0} &= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+z^2}} + \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+z^2}}\right] \\ &= -\frac{1}{2\pi}\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+z^2}} \end{aligned}$$

Therefore, the solution to Poisson's equation in the upper half-space with a Neumann boundary condition is

$$\begin{aligned} U(x,y,z) = \bar{U} - \frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty f(x_0,y_0,z_0)\left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right]dx_0\,dy_0\,dz_0 \\ - \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{h(x_0,y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2+z^2}}\,dx_0\,dy_0. \end{aligned}$$

If $f = 0$, then the solution reduces to

$$U(x,y,z) = \overline{U} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{h(x_0,y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2+z^2}}\,dx_0\,dy_0.$$

This answer is in disagreement with the one at the back of the book.

The boundary condition will now be verified. Differentiate U with respect to z.

$$\begin{aligned}
\frac{\partial U}{\partial z} &= \frac{\partial}{\partial z}\left[-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{h(x_0,y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2+z^2}}\,dx_0\,dy_0\right] \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial}{\partial z}\left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+z^2}}\right]h(x_0,y_0)\,dx_0\,dy_0 \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{-\frac{1}{2}\frac{2z}{[(x-x_0)^2+(y-y_0)^2+z^2]^{3/2}}\right\}h(x_0,y_0)\,dx_0\,dy_0 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z}{[(x-x_0)^2+(y-y_0)^2+z^2]^{3/2}}h(x_0,y_0)\,dx_0\,dy_0
\end{aligned}$$

Make the change of variables,

$$\begin{aligned}
x_0 - x &= r_0 z\cos\theta_0 \\
y_0 - y &= r_0 z\sin\theta_0.
\end{aligned}$$

The resulting Jacobian is

$$\frac{\partial(x_0,y_0)}{\partial(r_0,\theta_0)} = \begin{vmatrix}\frac{\partial x_0}{\partial r_0} & \frac{\partial x_0}{\partial \theta_0} \\ \frac{\partial y_0}{\partial r_0} & \frac{\partial y_0}{\partial \theta_0}\end{vmatrix} = \begin{vmatrix} z\cos\theta_0 & -r_0 z\sin\theta_0 \\ z\sin\theta_0 & r_0 z\cos\theta_0\end{vmatrix} = r_0 z^2\cos^2\theta_0 + r_0 z^2\sin^2\theta_0 = r_0 z^2,$$

which means $\partial U/\partial z$ becomes

$$\begin{aligned}
\frac{\partial U}{\partial z} &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{z}{[(-r_0 z\cos\theta_0)^2+(-r_0 z\sin\theta_0)^2+z^2]^{3/2}}h(x+r_0 z\cos\theta_0, y+r_0 z\sin\theta_0)r_0 z^2\,dr_0\,d\theta_0 \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{z^3}{[r_0^2 z^2(\cos^2\theta_0+\sin^2\theta_0)+z^2]^{3/2}}h(x+r_0 z\cos\theta_0, y+r_0 z\sin\theta_0)r_0\,dr_0\,d\theta_0 \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{z^3}{z^3(r_0^2+1)^{3/2}}h(x+r_0 z\cos\theta_0, y+r_0 z\sin\theta_0)r_0\,dr_0\,d\theta_0 \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{r_0}{(r_0^2+1)^{3/2}}h(x+r_0 z\cos\theta_0, y+r_0 z\sin\theta_0)\,dr_0\,d\theta_0.
\end{aligned}$$

Take the limit as $z \to 0$ now.

$$\begin{aligned}
\lim_{z\to 0}\frac{\partial U}{\partial z} &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{r_0}{(r_0^2+1)^{3/2}}h(x,y)\,dr_0\,d\theta_0 \\
&= \frac{h(x,y)}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{r_0}{(r_0^2+1)^{3/2}}\,dr_0\,d\theta_0 \\
&= h(x,y)
\end{aligned}$$