Exercise 1

Find all the three-dimensional plane waves; that is, all the solutions of the wave equation of the form $u(\mathbf{x}, t) = f(\mathbf{k} \cdot \mathbf{x} - ct)$, where $\mathbf{k}$ is a fixed vector and f is a function of one variable.

Solution

The three-dimensional wave equation is

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}).$$

Substitute the general form of a plane wave with $\mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}}$,

$$u(x, y, z, t) = f(k_x x + k_y y + k_z z - ct),$$

into the equation to determine f.

$$\begin{aligned} (-c)^2 f'' &= c^2[(k_x)^2 f'' + (k_y)^2 f'' + (k_z)^2 f''] \\ c^2 f'' &= c^2(k_x^2 + k_y^2 + k_z^2) f'' \end{aligned}$$

There are two ways this equation can be satisfied.

$$\begin{aligned} k_x^2 + k_y^2 + k_z^2 = 1 \quad &\text{or} \quad f''(k_x x + k_y y + k_z z - ct) = 0 \\ \sqrt{k_x^2 + k_y^2 + k_z^2} = 1 \quad &\text{or} \quad f'(k_x x + k_y y + k_z z - ct) = C_1 \\ |\mathbf{k}| = 1 \quad &\text{or} \quad f(k_x x + k_y y + k_z z - ct) = C_1(k_x x + k_y y + k_z z - ct) + C_2 \\ & \qquad f(\mathbf{k} \cdot \mathbf{x} - ct) = C_1(\mathbf{k} \cdot \mathbf{x} - ct) + C_2 \end{aligned}$$

Therefore, the three-dimensional plane waves are

$$u(\mathbf{x}, t) = C_1(\mathbf{k} \cdot \mathbf{x} - ct) + C_2,$$

but if it so happens that $|\mathbf{k}| = 1$, then $u(\mathbf{x}, t) = f(\mathbf{k} \cdot \mathbf{x} - ct)$, where f remains arbitrary.