Exercise 3

Verify that $(c^2t^2 - x^2 - y^2)^{-1/2}$ satisfies the two-dimensional wave equation except on the cone $\{x^2 + y^2 = c^2t^2\}$.

Solution

The two-dimensional wave equation is

$$u_{tt} = c^2(u_{xx} + u_{yy}).$$

Find all the second derivatives of $u(x, y, t) = (c^2t^2 - x^2 - y^2)^{-1/2}$.

$$\begin{aligned}
u_t &= -\frac{1}{2}(c^2t^2 - x^2 - y^2)^{-3/2}(2c^2t) \\
u_{tt} &= \frac{3}{4}(c^2t^2 - x^2 - y^2)^{-5/2}(2c^2t)^2 - \frac{1}{2}(c^2t^2 - x^2 - y^2)^{-3/2}(2c^2) \\
&= \frac{3c^4t^2}{(c^2t^2 - x^2 - y^2)^{5/2}} - \frac{c^2}{(c^2t^2 - x^2 - y^2)^{3/2}} \\
&= \frac{3c^4t^2 - c^2(c^2t^2 - x^2 - y^2)}{(c^2t^2 - x^2 - y^2)^{5/2}} \\
&= \frac{2c^4t^2 + c^2(x^2 + y^2)}{(c^2t^2 - x^2 - y^2)^{5/2}} \\
u_x &= -\frac{1}{2}(c^2t^2 - x^2 - y^2)^{-3/2}(-2x) \\
u_{xx} &= \frac{3}{4}(c^2t^2 - x^2 - y^2)^{-5/2}(-2x)^2 - \frac{1}{2}(c^2t^2 - x^2 - y^2)^{-3/2}(-2) \\
&= \frac{3x^2}{(c^2t^2 - x^2 - y^2)^{5/2}} + \frac{1}{(c^2t^2 - x^2 - y^2)^{3/2}} \\
&= \frac{3x^2 + c^2t^2 - x^2 - y^2}{(c^2t^2 - x^2 - y^2)^{5/2}} \\
&= \frac{2x^2 + c^2t^2 - y^2}{(c^2t^2 - x^2 - y^2)^{5/2}} \\
u_y &= -\frac{1}{2}(c^2t^2 - x^2 - y^2)^{-3/2}(-2y) \\
u_{yy} &= \frac{3}{4}(c^2t^2 - x^2 - y^2)^{-5/2}(-2y)^2 - \frac{1}{2}(c^2t^2 - x^2 - y^2)^{-3/2}(-2) \\
&= \frac{3y^2}{(c^2t^2 - x^2 - y^2)^{5/2}} + \frac{1}{(c^2t^2 - x^2 - y^2)^{3/2}} \\
&= \frac{3y^2 + c^2t^2 - x^2 - y^2}{(c^2t^2 - x^2 - y^2)^{5/2}} \\
&= \frac{2y^2 + c^2t^2 - x^2}{(c^2t^2 - x^2 - y^2)^{5/2}}
\end{aligned}$$

Substitute these formulas into the wave equation to check whether $u(x,y,t)$ is indeed a solution.

$$\begin{aligned}
\frac{2c^4t^2+c^2(x^2+y^2)}{(c^2t^2-x^2-y^2)^{5/2}} &\stackrel{?}{=} c^2\left[\frac{2x^2+c^2t^2-y^2}{(c^2t^2-x^2-y^2)^{5/2}}+\frac{2y^2+c^2t^2-x^2}{(c^2t^2-x^2-y^2)^{5/2}}\right] \\
&\stackrel{?}{=} c^2\left[\frac{2x^2+c^2t^2-y^2+2y^2+c^2t^2-x^2}{(c^2t^2-x^2-y^2)^{5/2}}\right] \\
&\stackrel{?}{=} c^2\left[\frac{x^2+y^2+2c^2t^2}{(c^2t^2-x^2-y^2)^{5/2}}\right] \\
&= \frac{2c^4t^2+c^2(x^2+y^2)}{(c^2t^2-x^2-y^2)^{5/2}}
\end{aligned}$$

Therefore, $u(x,y,t)=(c^2t^2-x^2-y^2)^{-1/2}$ satisfies the wave equation except when $c^2t^2=x^2+y^2$.