Exercise 5

Prove the principle of causality in two dimensions.

Solution

The principle of causality for the wave equation in two dimensions states that the solution u at a particular point in space-time (x_0, y_0, t_0) is completely determined by the initial conditions, $u(x, y, 0) = \phi(x, y)$ and $u_t(x, y, 0) = \psi(x, y)$, in the disk $(x - x_0)^2 + (y - y_0)^2 \leq c^2 t_0^2$. Consider first the wave equation in one dimension.

$$u_{tt} - c^2 u_{xx} = 0$$

Comparing this to the general form of a linear second-order PDE, $Au_{tt} + Bu_{tx} + Cu_{xx} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The equations for the characteristic curves of the wave equation are given by

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC}) \\ &= \frac{1}{2}(\pm\sqrt{4c^2}) \\ &= \pm c. \end{aligned}$$

Solving these ODEs results in two families of characteristic curves, each with its own characteristic coordinate.

$$\frac{dx}{dt} = \pm c \quad \rightarrow \quad \begin{cases} x = ct + \xi \\ x = -ct + \eta \end{cases}$$

The characteristic curves in the xt-plane are lines. The equations for the ones passing through a particular point (x_0, t_0) are

$$x - x_0 = \pm c(t - t_0).$$

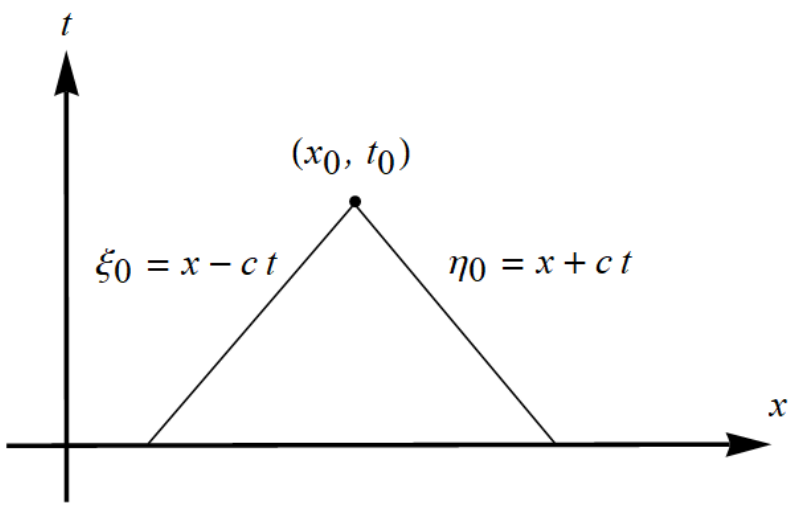

Figure 1: Each point in the xt-plane has a characteristic triangle associated with it.

Now introduce a second spatial dimension y. Rotating the characteristic lines about the t-axis results in characteristic cones for the two-dimensional wave equation.

$$\begin{aligned} |\mathbf{x} - \mathbf{x}_0| &= \pm c(t - t_0) \\ \sqrt{(x - x_0)^2 + (y - y_0)^2} &= \pm c(t - t_0) \\ (x - x_0)^2 + (y - y_0)^2 &= c^2(t - t_0)^2 \end{aligned} \tag{1}$$

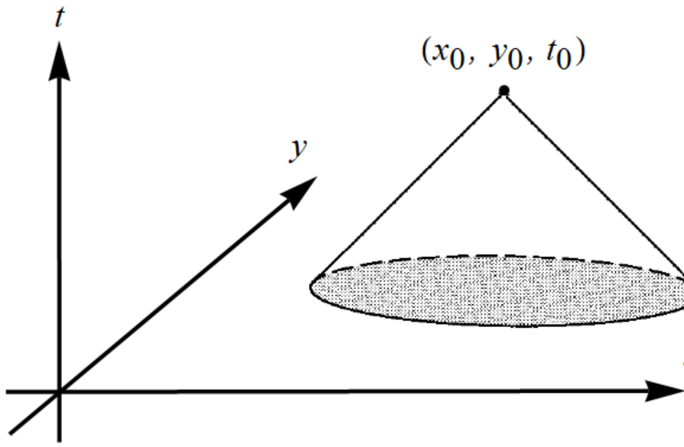

Figure 2: Each point in three-dimensional space-time has a characteristic cone associated with it. The shaded disk lies in the xy-plane, that is, when $t = 0$: $(x - x_0)^2 + (y - y_0)^2 \leq c^2 t_0^2$.

The two-dimensional wave equation is

$$u_{tt} = c^2(u_{xx} + u_{yy}).$$

Multiply both sides by u_t.

$$u_t u_{tt} = c^2(u_t u_{xx} + u_t u_{yy})$$

Rewrite each term in the equation as follows.

$$\frac{1}{2}\frac{\partial}{\partial t}(u_t^2) = c^2\left[\frac{\partial}{\partial x}(u_t u_x) - u_{xt}u_x + \frac{\partial}{\partial y}(u_t u_y) - u_{yt}u_y\right]$$

$$\frac{1}{2}\frac{\partial}{\partial t}(u_t^2) = c^2\left[\frac{\partial}{\partial x}(u_t u_x) - \frac{1}{2}\frac{\partial}{\partial t}(u_x^2) + \frac{\partial}{\partial y}(u_t u_y) - \frac{1}{2}\frac{\partial}{\partial t}(u_y^2)\right]$$

Bring the temporal derivatives to the left side.

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}(u_t^2) + c^2\left[\frac{1}{2}\frac{\partial}{\partial t}(u_x^2) + \frac{1}{2}\frac{\partial}{\partial t}(u_y^2)\right] &= c^2\left[\frac{\partial}{\partial x}(u_t u_x) + \frac{\partial}{\partial y}(u_t u_y)\right] \\ \frac{1}{2}\frac{\partial}{\partial t}\left[u_t^2 + c^2(u_x^2 + u_y^2)\right] &= c^2\frac{\partial}{\partial x}(u_t u_x) + c^2\frac{\partial}{\partial y}(u_t u_y) \\ \frac{\partial}{\partial t}\left[\frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\right] &= \frac{\partial}{\partial x}(c^2 u_t u_x) + \frac{\partial}{\partial y}(c^2 u_t u_y) \end{aligned}$$

Bring the two spatial derivatives to the left side.

$$\frac{\partial}{\partial x}(-c^2 u_t u_x) + \frac{\partial}{\partial y}(-c^2 u_t u_y) + \frac{\partial}{\partial t}\left[\frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\right] = 0$$

Integrate both sides over the volume of the frustum W illustrated in Figure 3.

$$\iiint_W \left\{\frac{\partial}{\partial x}(-c^2 u_t u_x) + \frac{\partial}{\partial y}(-c^2 u_t u_y) + \frac{\partial}{\partial t}\left[\frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\right]\right\} dV = \iiint_W 0\, dV$$

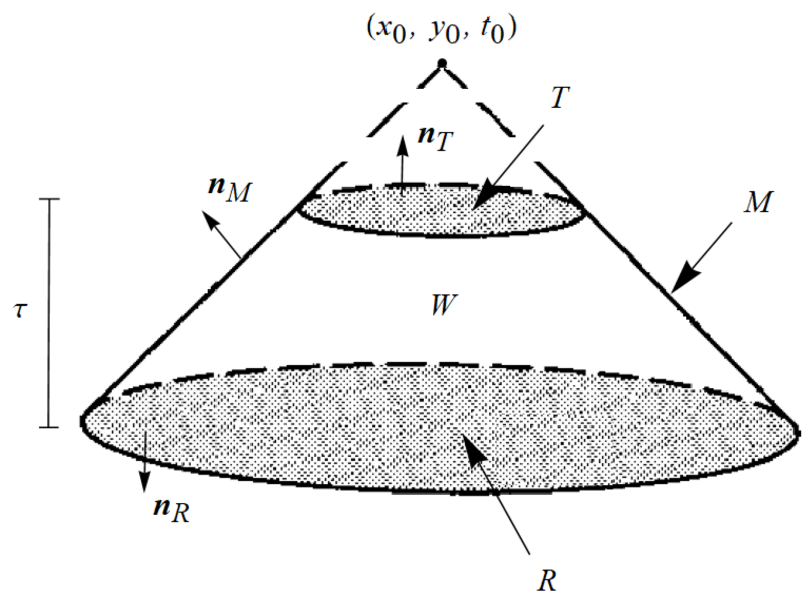

Figure 3: This is an illustration of the frustum $W = \{(x, y, t) \mid (x-x_0)^2+(y-y_0)^2 \leq c^2(t_0-t)^2,\ 0 \leq t \leq \tau\}$. Its boundary consists of the bottom $R = \{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 \leq c^2 t_0^2\}$, the top $T = \{(x, y) \mid (x-x_0)^2+(y-y_0)^2 \leq c^2(t_0-\tau)^2\}$, and the mantle $M = \{(x, y, t) \mid (x-x_0)^2+(y-y_0)^2 = c^2(t_0 - t)^2,\ 0 \leq t \leq \tau\}$. τ represents the adjustable (temporal) height of the frustum: $0 < \tau < t_0$. In addition, $\mathbf{n}_R$, $\mathbf{n}_T$, and $\mathbf{n}_M$ are the outward unit vectors normal to the respective faces.

The integrand in curly braces is the divergence of a vector (treating t as if it were z).

$$\iiint_W \nabla \cdot \left\langle -c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\right\rangle dV = 0$$

Apply the divergence theorem to turn this volume integral into a surface integral over the frustum's boundary bdy W.

$$\oiint_{\text{bdy } W} \left\langle -c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\right\rangle \cdot \mathbf{n}\, dS = 0$$

$\mathbf{n}$ is the outward unit vector normal to the frustum. Split up the closed surface integral into a surface integral over each face of the frustum.

$$\iint_R \left\langle -c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}(u_t^2 + c^2|\nabla u|^2) \right\rangle \cdot \mathbf{n}_R \, dS + \iint_T \left\langle -c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}(u_t^2 + c^2|\nabla u|^2) \right\rangle \cdot \mathbf{n}_T \, dS + \iint_M \left\langle -c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}(u_t^2 + c^2|\nabla u|^2) \right\rangle \cdot \mathbf{n}_M \, dS = 0 \qquad (2)$$

Inspecting the frustum in Figure 3, the outward normal unit vectors are

$$\mathbf{n}_R = \langle 0, 0, -1 \rangle$$
$$\mathbf{n}_T = \langle 0, 0, 1 \rangle$$
$$\mathbf{n}_M = \frac{\nabla \phi}{|\nabla \phi|},$$

where $\phi = \phi(x, y, t)$ is a level curve of the characteristic cone in equation (1).

$$\phi(x, y, t) = (x - x_0)^2 + (y - y_0)^2 - c^2(t - t_0)^2 = 0$$

As a result,

$$\begin{aligned}
\mathbf{n}_M &= \frac{\langle 2(x-x_0), 2(y-y_0), -2c^2(t-t_0)\rangle}{\sqrt{[2(x-x_0)]^2+[2(y-y_0)]^2+[-2c^2(t-t_0)]^2}} \\
&= \frac{\langle x-x_0, y-y_0, c^2(t_0-t)\rangle}{\sqrt{(x-x_0)^2+(y-y_0)^2+c^4(t-t_0)^2}} \\
&= \left\langle \frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+c^4(t-t_0)^2}}, \frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+c^4(t-t_0)^2}}, \frac{c^2(t_0-t)}{\sqrt{(x-x_0)^2+(y-y_0)^2+c^4(t-t_0)^2}} \right\rangle.
\end{aligned}$$

Use equation (1) to eliminate $c^2(t-t_0)^2$ in the first two components and to eliminate $(x-x_0)^2+(y-y_0)^2$ in the third component.

$$\begin{aligned}
&= \left\langle \frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+c^2[(x-x_0)^2+(y-y_0)^2]}}, \frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+c^2[(x-x_0)^2+(y-y_0)^2]}}, \frac{c^2(t_0-t)}{\sqrt{c^2(t-t_0)^2+c^4(t-t_0)^2}} \right\rangle \\
&= \left\langle \frac{x-x_0}{\sqrt{(1+c^2)[(x-x_0)^2+(y-y_0)^2]}}, \frac{y-y_0}{\sqrt{(1+c^2)[(x-x_0)^2+(y-y_0)^2]}}, \frac{c^2(t_0-t)}{\sqrt{c^2(1+c^2)(t-t_0)^2}} \right\rangle \\
&= \frac{1}{\sqrt{1+c^2}} \left\langle \frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}, \frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}, \frac{c(t_0-t)}{\sqrt{(t-t_0)^2}} \right\rangle \\
&= \frac{c}{\sqrt{1+c^2}} \left\langle \frac{x-x_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2}}, \frac{y-y_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2}}, \frac{t_0-t}{|t-t_0|} \right\rangle
\end{aligned}$$

Within the frustum $t < t_0$, so $|t-t_0| = -(t-t_0) = t_0 - t$.

$$= \frac{c}{\sqrt{1+c^2}} \left\langle \frac{x-x_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2}}, \frac{y-y_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2}}, 1 \right\rangle$$

With these unit vectors equation (2) becomes

$$\begin{aligned}
&\iint_R \left\langle -c^2u_tu_x, -c^2u_tu_y, \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\right\rangle \cdot \langle 0,0,-1\rangle\, dS \\
&\quad + \iint_T \left\langle -c^2u_tu_x, -c^2u_tu_y, \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\right\rangle \cdot \langle 0,0,1\rangle\, dS \\
&\qquad + \iint_M \left\langle -c^2u_tu_x, -c^2u_tu_y, \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\right\rangle \cdot \frac{c}{\sqrt{1+c^2}} \left\langle \frac{x-x_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2}}, \frac{y-y_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2}}, 1\right\rangle dS = 0 \\
&\iint_R \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)(-1)\, dS \\
&\quad + \iint_T \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\, dS \\
&\qquad + \frac{c}{\sqrt{1+c^2}} \iint_M \left[(-c^2u_tu_x)\frac{x-x_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2}} + (-c^2u_tu_y)\frac{y-y_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2}} + \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\right] dS = 0 \\
&-\iint_R \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\, dS \\
&\quad + \iint_T \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\, dS \\
&\qquad + \frac{c}{\sqrt{1+c^2}} \iint_M \left\{(-cu_t)\left[u_x\frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}} + u_y\frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right] + \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\right\} dS = 0. \qquad (3)
\end{aligned}$$

The integral over the mantle M will now be shown to be positive (or zero).

$$\frac{c}{\sqrt{1+c^2}}\iint\limits_M \left[(-cu_t)\,\langle u_x,u_y\rangle\cdot\left\langle\frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}},\frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right\rangle+\frac{1}{2}(u_t^2+c^2|\nabla u|^2)\right]dS$$

$$=\frac{c}{\sqrt{1+c^2}}\iint\limits_M \left[(-cu_t)\nabla u\cdot\frac{\langle x-x_0,y-y_0\rangle}{\sqrt{(x-x_0)^2+(y-y_0)^2}}+\frac{1}{2}(u_t^2+c^2|\nabla u|^2)\right]dS$$

If we let $\boldsymbol{\mathscr{r}}=\langle x-x_0,y-y_0\rangle$, then the dot product can be interpreted as a directional derivative in the direction of $\hat{\boldsymbol{\mathscr{r}}}$: $\nabla u\cdot\hat{\boldsymbol{\mathscr{r}}}=\partial u/\partial\mathscr{r}$.

$$=\frac{c}{\sqrt{1+c^2}}\iint\limits_M \left[(-cu_t)\nabla u\cdot\hat{\boldsymbol{\mathscr{r}}}+\frac{1}{2}(u_t^2+c^2|\nabla u|^2)\right]dS$$

$$=\frac{c}{\sqrt{1+c^2}}\iint\limits_M \left[-cu_tu_{\mathscr{r}}+\frac{1}{2}(u_t^2+c^2|\nabla u|^2)\right]dS$$

$$=\frac{c}{\sqrt{1+c^2}}\iint\limits_M \left[\frac{1}{2}(u_t^2-2cu_tu_{\mathscr{r}})+\frac{1}{2}c^2|\nabla u|^2\right]dS$$

Complete the square.

$$=\frac{c}{\sqrt{1+c^2}}\iint\limits_M \left[\frac{1}{2}(u_t^2-2cu_tu_{\mathscr{r}}+c^2u_{\mathscr{r}}^2)+\frac{1}{2}c^2|\nabla u|^2-\frac{1}{2}c^2u_{\mathscr{r}}^2\right]dS$$

$$=\frac{c}{\sqrt{1+c^2}}\iint\limits_M \left[\frac{1}{2}(u_t-cu_{\mathscr{r}})^2+\frac{1}{2}c^2(|\nabla u|^2-u_{\mathscr{r}}^2)\right]dS$$

$$=\frac{c}{\sqrt{1+c^2}}\iint\limits_M \left[\frac{1}{2}(u_t-cu_{\mathscr{r}})^2+\frac{1}{2}c^2(\nabla u-u_{\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}})\cdot(\nabla u-u_{\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}})\right]dS$$

$$=\frac{c}{\sqrt{1+c^2}}\iint\limits_M \left[\frac{1}{2}(u_t-cu_{\mathscr{r}})^2+\frac{1}{2}c^2|\nabla u-u_{\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}}|^2\right]dS$$

The integrand is a sum of squared terms, so the integral over M is positive (it could be equal to zero if u is a constant). Bringing the integral over R to the right side, equation (3) becomes

$$\iint\limits_T \frac{1}{2}(u_t^2+c^2|\nabla u|^2)\,dS+\frac{c}{\sqrt{1+c^2}}\iint\limits_M \left[\frac{1}{2}(u_t-cu_{\mathscr{r}})^2+\frac{1}{2}c^2|\nabla u-u_{\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}}|^2\right]dS=\iint\limits_R \frac{1}{2}(u_t^2+c^2|\nabla u|^2)\,dS$$

from which we conclude that

$$\iint\limits_T \frac{1}{2}(u_t^2+c^2|\nabla u|^2)\,dS\le\iint\limits_R \frac{1}{2}(u_t^2+c^2|\nabla u|^2)\,dS.$$

The R face of the frustum lies in the $t=0$ plane, so u and u_t can be replaced by ϕ and ψ, respectively.

$$\iint\limits_T \frac{1}{2}(u_t^2+c^2|\nabla u|^2)\,dS\le\iint\limits_R \frac{1}{2}(\psi^2+c^2|\nabla\phi|^2)\,dS$$

Substitute the formulas for T and R and note that dS is $dx\,dy$ on the top and bottom faces of the frustum.

$$\iint\limits_{(x-x_0)^2+(y-y_0)^2\leq c^2(t_0-\tau)^2} \frac{1}{2}(u_t^2+c^2|\nabla u|^2)\,dx\,dy \leq \iint\limits_{(x-x_0)^2+(y-y_0)^2\leq c^2t_0^2} \frac{1}{2}(\psi^2+c^2|\nabla\phi|^2)\,dx\,dy$$

These double integrals can be written explicitly by using polar coordinates (r, θ).

$$x - x_0 = r\cos\theta$$
$$y - y_0 = r\sin\theta$$

Therefore, the principle of causality for the wave equation in two dimensions is

$$\int_0^{2\pi}\int_0^{c(t_0-\tau)} \frac{1}{2}(u_t^2+c^2|\nabla u|^2)r\,dr\,d\theta \leq \int_0^{2\pi}\int_0^{ct_0} \frac{1}{2}(\psi^2+c^2|\nabla\phi|^2)r\,dr\,d\theta,$$

where $0<\tau<t_0$. The energy within a disk of radius $c(t_0-t)$ is

$$\mathscr{E}(t) = \int_0^{2\pi}\int_0^{c(t_0-t)} \frac{1}{2}(u_t^2+c^2|\nabla u|^2)r\,dr\,d\theta,$$

so the causality principle can be expressed compactly as

$$\mathscr{E}(\tau) \leq \mathscr{E}(0).$$

This implies that the energy at a particular point (x_0, y_0, t_0) cannot be larger than what is initially within the disk $(x-x_0)^2+(y-y_0)^2 \leq c^2t_0^2$. It follows that c is the maximum possible speed a solution of the wave equation can travel because otherwise points in the xy-plane outside the disk would also contribute to the energy at (x_0, y_0, t_0).