Exercise 6

(a) Derive the conservation of energy for the wave equation in a domain D with homogeneous Dirichlet or Neumann boundary conditions.

(b) What about the Robin condition?

Solution

The wave equation is

$$u_{tt} = c^2\nabla^2 u.$$

Expand the Laplacian operator in Cartesian coordinates for the time being.

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$$

Multiply both sides by u_t.

$$u_t u_{tt} = c^2(u_t u_{xx} + u_t u_{yy} + u_t u_{zz})$$

Rewrite each term in this equation.

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}(u_t^2) &= c^2\left[\frac{\partial}{\partial x}(u_t u_x) - u_{xt}u_x + \frac{\partial}{\partial y}(u_t u_y) - u_{yt}u_y + \frac{\partial}{\partial z}(u_t u_z) - u_{zt}u_z\right] \\ &= c^2\left[\frac{\partial}{\partial x}(u_t u_x) - \frac{1}{2}\frac{\partial}{\partial t}(u_x^2) + \frac{\partial}{\partial y}(u_t u_y) - \frac{1}{2}\frac{\partial}{\partial t}(u_y^2) + \frac{\partial}{\partial z}(u_t u_z) - \frac{1}{2}\frac{\partial}{\partial t}(u_z^2)\right] \end{aligned}$$

Bring all time derivatives to the left side.

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}(u_t^2) + c^2\left[\frac{1}{2}\frac{\partial}{\partial t}(u_x^2) + \frac{1}{2}\frac{\partial}{\partial t}(u_y^2) + \frac{1}{2}\frac{\partial}{\partial t}(u_z^2)\right] &= c^2\left[\frac{\partial}{\partial x}(u_t u_x) + \frac{\partial}{\partial y}(u_t u_y) + \frac{\partial}{\partial z}(u_t u_z)\right] \\ \frac{1}{2}\frac{\partial}{\partial t}\left[u_t^2 + c^2(u_x^2 + u_y^2 + u_z^2)\right] &= c^2\left[\frac{\partial}{\partial x}(u_t u_x) + \frac{\partial}{\partial y}(u_t u_y) + \frac{\partial}{\partial z}(u_t u_z)\right] \end{aligned}$$

Reintroduce vector operators into the equation.

$$\frac{1}{2}\frac{\partial}{\partial t}\left(u_t^2 + c^2|\nabla u|^2\right) = c^2\nabla\cdot(u_t\nabla u)$$

Integrate both sides over the domain's volume.

$$\iiint_D \frac{1}{2}\frac{\partial}{\partial t}\left(u_t^2 + c^2|\nabla u|^2\right)dV = \iiint_D c^2\nabla\cdot(u_t\nabla u)\,dV$$

Pull the time derivative in front of the integral on the left side, and bring the constant in front of the integral on the right side. The volume integral wipes out the spatial variables, so the time derivative is a total derivative in front of the integral.

$$\frac{d}{dt}\iiint_D \frac{1}{2}\left(u_t^2 + c^2|\nabla u|^2\right)dV = c^2\iiint_D \nabla\cdot(u_t\nabla u)\,dV$$

The triple integral on the left side is defined to be the energy E.

$$\frac{dE}{dt} = c^2\iiint_D \nabla\cdot(u_t\nabla u)\,dV$$

Apply the divergence theorem to the remaining triple integral to turn it into a surface integral over the domain's boundary bdy D.

$$\frac{dE}{dt} = c^2 \oiint_{\text{bdy } D} u_t \nabla u \cdot \hat{\mathbf{n}}\, dS, \tag{1}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the domain's boundary pointing outward.

The Dirichlet Boundary Condition

For a domain in space that has a homogeneous Dirichlet boundary condition, u vanishes on the boundary.

$$u = 0 \quad \text{on bdy } D$$

Differentiate both sides of the boundary condition with respect to t.

$$u_t = 0 \quad \text{on bdy } D$$

Consequently, equation (1) reduces to

$$\frac{dE}{dt} = 0,$$

which implies that energy remains constant in time. Therefore, conservation of energy holds in this case.

The Neumann Boundary Condition

For a domain in space that has a homogeneous Neumann boundary condition, the directional derivative of u in the outward normal direction vanishes.

$$\frac{\partial u}{\partial n} = \nabla u \cdot \hat{\mathbf{n}} = 0 \quad \text{on bdy } D$$

Consequently, equation (1) reduces to

$$\frac{dE}{dt} = 0,$$

which implies that energy remains constant in time. Therefore, conservation of energy holds in this case.

The Robin Boundary Condition

For a domain in space that has a homogeneous Robin boundary condition, the directional derivative of u in the outward normal direction is proportional to u.

$$\frac{\partial u}{\partial n} = \nabla u \cdot \hat{\mathbf{n}} = au \quad \text{on bdy } D,$$

where a is a proportionality constant. Consequently, equation (1) becomes

$$\frac{dE}{dt} = c^2 \oiint_{\text{bdy } D} u_t(au)\, dS = c^2 \oiint_{\text{bdy } D} \frac{a}{2}\frac{\partial}{\partial t}(u^2)\, dS = a\frac{c^2}{2}\frac{d}{dt} \oiint_{\text{bdy } D} u^2\, dS,$$

which implies that the energy grows in time if a is positive and decays in time if a is negative. Therefore, conservation of energy does not hold in this case.