Exercise 7

For the boundary condition $\partial u/\partial n + b\,\partial u/\partial t = 0$ with $b > 0$, show that the energy defined by (6) decreases.

Solution

The wave equation in a three-dimensional domain D is

$$u_{tt} = c^2 \nabla^2 u.$$

Expand the Laplacian operator in Cartesian coordinates for the time being.

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$$

Multiply both sides by u_t.

$$u_t u_{tt} = c^2(u_t u_{xx} + u_t u_{yy} + u_t u_{zz})$$

Rewrite each term in this equation.

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}(u_t^2) &= c^2\left[\frac{\partial}{\partial x}(u_t u_x) - u_{xt}u_x + \frac{\partial}{\partial y}(u_t u_y) - u_{yt}u_y + \frac{\partial}{\partial z}(u_t u_z) - u_{zt}u_z\right] \\ &= c^2\left[\frac{\partial}{\partial x}(u_t u_x) - \frac{1}{2}\frac{\partial}{\partial t}(u_x^2) + \frac{\partial}{\partial y}(u_t u_y) - \frac{1}{2}\frac{\partial}{\partial t}(u_y^2) + \frac{\partial}{\partial z}(u_t u_z) - \frac{1}{2}\frac{\partial}{\partial t}(u_z^2)\right] \end{aligned}$$

Bring all time derivatives to the left side.

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}(u_t^2) + c^2\left[\frac{1}{2}\frac{\partial}{\partial t}(u_x^2) + \frac{1}{2}\frac{\partial}{\partial t}(u_y^2) + \frac{1}{2}\frac{\partial}{\partial t}(u_z^2)\right] &= c^2\left[\frac{\partial}{\partial x}(u_t u_x) + \frac{\partial}{\partial y}(u_t u_y) + \frac{\partial}{\partial z}(u_t u_z)\right] \\ \frac{1}{2}\frac{\partial}{\partial t}\left[u_t^2 + c^2(u_x^2 + u_y^2 + u_z^2)\right] &= c^2\left[\frac{\partial}{\partial x}(u_t u_x) + \frac{\partial}{\partial y}(u_t u_y) + \frac{\partial}{\partial z}(u_t u_z)\right] \end{aligned}$$

Reintroduce vector operators into the equation.

$$\frac{1}{2}\frac{\partial}{\partial t}\left(u_t^2 + c^2|\nabla u|^2\right) = c^2 \nabla \cdot (u_t \nabla u)$$

Integrate both sides over the domain's volume.

$$\iiint_D \frac{1}{2}\frac{\partial}{\partial t}\left(u_t^2 + c^2|\nabla u|^2\right) dV = \iiint_D c^2 \nabla \cdot (u_t \nabla u)\, dV$$

Pull the time derivative in front of the integral on the left side, and bring the constant in front of the integral on the right side. The volume integral wipes out the spatial variables, so the time derivative is a total derivative in front of the integral.

$$\frac{d}{dt}\iiint_D \frac{1}{2}\left(u_t^2 + c^2|\nabla u|^2\right) dV = c^2 \iiint_D \nabla \cdot (u_t \nabla u)\, dV$$

The triple integral on the left side is defined to be the energy E—equation (6) in the textbook.

$$\frac{dE}{dt} = c^2 \iiint_D \nabla \cdot (u_t \nabla u)\, dV$$

Apply the divergence theorem to the remaining triple integral to turn it into a surface integral over the domain's boundary bdy D.

$$\frac{dE}{dt} = c^2 \oiint_{\text{bdy } D} u_t \nabla u \cdot \hat{\mathbf{n}} \, dS, \tag{1}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the domain's boundary pointing outward. Note that $\partial u/\partial n$ and $\nabla u \cdot \hat{\mathbf{n}}$ both represent the directional derivative of u in the outward normal direction. The prescribed boundary condition yields

$$\frac{\partial u}{\partial n} + b\frac{\partial u}{\partial t} = 0 \quad \rightarrow \quad \nabla u \cdot \hat{\mathbf{n}} + bu_t = 0 \quad \rightarrow \quad \nabla u \cdot \hat{\mathbf{n}} = -bu_t \quad \text{on bdy } D.$$

Consequently, equation (1) becomes

$$\frac{dE}{dt} = c^2 \oiint_{\text{bdy } D} u_t(-bu_t) \, dS = -bc^2 \oiint_{\text{bdy } D} u_t^2 \, dS.$$

The integral, c^2, and b are all positive; therefore, the energy decreases in time.