Exercise 1

Prove that $\Delta(\overline{u}) = (\overline{\Delta u})$ for any function; that is, the laplacian of the average is the average of the laplacian. (*Hint:* Write Δu in spherical coordinates and show that the angular terms have zero average on spheres centered at the origin.)

Solution

$\overline{u} = \overline{u}(r,t)$ is defined to be the average of u over a sphere of radius r,

$$\overline{u} = \frac{\iint u\,dS}{\iint dS} = \frac{\int_0^{2\pi}\int_0^{\pi} u(r,\phi,\theta,t)r^2\sin\theta\,d\theta\,d\phi}{4\pi r^2} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(r,\phi,\theta,t)\sin\theta\,d\theta\,d\phi,$$

and the Laplacian operator expands in spherical coordinates as

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2},$$

where θ is the angle from the polar axis. Note that $\overline{u}$ is only a function of r and t because the double integral over the surface wipes out the ϕ and θ variables. We have

$$\begin{aligned}
\Delta(\overline{u}) &= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\overline{u} \\
&= \frac{\partial^2}{\partial r^2}\overline{u} + \frac{2}{r}\frac{\partial}{\partial r}\overline{u} + \underbrace{\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\overline{u}}_{=0} + \underbrace{\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\overline{u}}_{=0} + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\overline{u}}_{=0} \\
&= \frac{\partial^2}{\partial r^2}\left[\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(r,\phi',\theta',t)\sin\theta'\,d\theta'\,d\phi'\right] + \frac{2}{r}\frac{\partial}{\partial r}\left[\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(r,\phi',\theta',t)\sin\theta'\,d\theta'\,d\phi'\right] \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\partial^2 u}{\partial r^2}\sin\theta'\,d\theta'\,d\phi' + \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{2}{r}\frac{\partial u}{\partial r}\sin\theta'\,d\theta'\,d\phi' \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)\sin\theta'\,d\theta'\,d\phi'.
\end{aligned}$$

Primes were put on the dummy integration variables to distinguish them from the spatial variables. Now we will show that $\overline{\Delta u}$ yields the same expression.

$$\begin{aligned}
\overline{\Delta u} &= \frac{\iint \Delta u\,dS}{\iint dS} \\
&= \frac{\int_0^{2\pi}\int_0^{\pi}\Delta u\,r^2\sin\theta\,d\theta\,d\phi}{4\pi r^2} \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\Delta u\sin\theta\,d\theta\,d\phi
\end{aligned}$$

$$\begin{aligned}\overline{\Delta u} &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right)\sin\theta\,d\theta\,d\phi \\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)\sin\theta\,d\theta\,d\phi \\ &\qquad + \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right)\sin\theta\,d\theta\,d\phi\end{aligned}$$

All that's left to do is to show that this second double integral vanishes.

$$\begin{aligned}&\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right)\sin\theta\,d\theta\,d\phi \\ &\qquad = \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi}\left(\sin\theta\frac{\partial^2 u}{\partial \theta^2} + \cos\theta\frac{\partial u}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial^2 u}{\partial \phi^2}\right)d\theta\,d\phi \\ &\qquad = \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 u}{\partial \phi^2}\right]d\theta\,d\phi \\ &\qquad = \frac{1}{4\pi r^2}\left[\int_0^{2\pi}\int_0^{\pi}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)d\theta\,d\phi + \int_0^{2\pi}\int_0^{\pi}\frac{1}{\sin\theta}\frac{\partial^2 u}{\partial \phi^2}\,d\theta\,d\phi\right] \\ &\qquad = \frac{1}{4\pi r^2}\left[\int_0^{2\pi}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)\bigg|_0^{\pi}d\phi + \int_0^{\pi}\frac{1}{\sin\theta}\left(\int_0^{2\pi}\frac{\partial^2 u}{\partial \phi^2}\,d\phi\right)d\theta\right] \\ &\qquad = \frac{1}{4\pi r^2}\left[\int_0^{2\pi}\left(\sin\pi\frac{\partial u}{\partial \theta}\bigg|_{\theta=\pi} - \sin 0\frac{\partial u}{\partial \theta}\bigg|_{\theta=0}\right)d\phi + \int_0^{\pi}\frac{1}{\sin\theta}\left(\frac{\partial u}{\partial \phi}\bigg|_{\phi=2\pi} - \frac{\partial u}{\partial \phi}\bigg|_{\phi=0}\right)d\theta\right] \\ &\qquad = 0\end{aligned}$$

The last step follows because $\partial u/\partial\phi$ has the same value at $\phi = 0$ as it does at $\phi = 2\pi$. As a result,

$$\overline{\Delta u} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)\sin\theta\,d\theta\,d\phi.$$

Therefore,

$$\Delta(\overline{u}) = (\overline{\Delta u}).$$