Exercise 2

Verify that (3) is correct in the case of the example $u(x, y, z, t) \equiv t$.

Solution

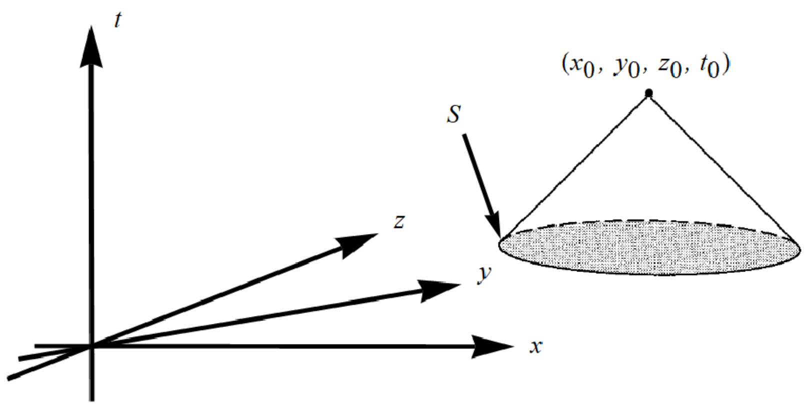
All the second derivatives of $u(x, y, z, t) = t$ are equal to zero, so u is a solution to the three-dimensional wave equation,

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}).$$

We will verify that equation (3) in the textbook, [TYPO: $\mathbf{x}$ should be $\mathbf{x}_0$]

$$u(\mathbf{x}, t_0) = \frac{1}{4\pi c^2 t_0} \iint_S \psi(\mathbf{x})\, dS + \frac{\partial}{\partial t_0} \left[\frac{1}{4\pi c^2 t_0} \iint_S \phi(\mathbf{x})\, dS \right], \tag{3}$$

gives this solution. Note that S is the sphere centered at (x_0, y_0, z_0) with radius ct_0: $S = \{(x, y, z) \mid (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = c^2 t_0^2\}$. It is the boundary of the shaded hyperdisk in the xyz-plane, and it is a sphere in xyz-space.



The x-, y-, and z-axes are perpendicular to each other in addition to the t-axis. ϕ and ψ are the initial conditions of u.

$$\begin{aligned} u(x, y, z, 0) &= \phi(x, y, z) = 0 \\ u_t(x, y, z, 0) &= \psi(x, y, z) = 1 \end{aligned}$$

Substitute these functions into the formula.

$$\begin{aligned} u(\mathbf{x}_0, t_0) &= \frac{1}{4\pi c^2 t_0} \iint_S (1)\, dS + \frac{\partial}{\partial t_0} \left[\frac{1}{4\pi c^2 t_0} \iint_S (0)\, dS \right] \\ &= \frac{1}{4\pi c^2 t_0} \iint_S dS \\ &= \frac{1}{4\pi c^2 t_0} [4\pi (ct_0)^2] \\ &= t_0 \end{aligned}$$

Therefore, equation (3) gives the correct answer for this example.