Exercise 7

(a) Solve the wave equation in three dimensions for $t > 0$ with the initial conditions $\phi(\mathbf{x}) = A$ for $|\mathbf{x}| < \rho$, $\phi(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho$, and $\psi|\mathbf{x}| \equiv 0$, where A is a constant. (This is somewhat like the plucked string.) (*Hint:* Differentiate the solution in Exercise 6(b).)

(b) Sketch the regions in space-time that illustrate your answer. Where does the solution have jump discontinuities?

(c) Let $|\mathbf{x}_0| < \rho$. Ride the wave along a light ray emanating from $(\mathbf{x}_0, 0)$. That is, look at $u(\mathbf{x}_0 + t\mathbf{v}, t)$ where $|\mathbf{v}| = c$. Prove that

$$t \cdot u(\mathbf{x}_0 + t\mathbf{v}, t) \text{ converges as } t \to \infty.$$

Solution

Part (a)

The solution of the three-dimensional wave equation in space subject to two initial conditions,

$$\begin{aligned} &u_{tt} = c^2\nabla^2 u, \quad -\infty < x, y, z < \infty,\ t > 0 \\ &u(x, y, z, 0) = \alpha(x, y, z) \\ &u_t(x, y, z, 0) = \beta(x, y, z), \end{aligned}$$

is given by the formula of Kirchhoff and Poisson.

$$u(x, y, z, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \alpha(x_0, y_0, z_0)\, dS_0\right] + \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \beta(x_0, y_0, z_0)\, dS_0$$

In particular, we wish to solve the initial value problem when

$$\alpha(x, y, z) = \begin{cases} A & r < \rho \\ 0 & r > \rho \end{cases} \quad \text{and} \quad \beta(x, y, z) = 0,$$

where $r = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$. With these initial conditions, the previous formula simplifies to

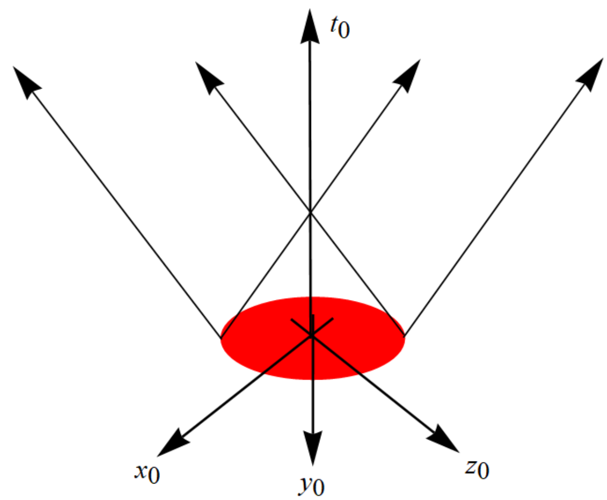
$$u(x, y, z, t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t} \iint\limits_{Q\cap T} A\, dS_0\right),$$

where

$$\begin{aligned} Q &= \{(x_0, y_0, z_0) \mid (x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 = c^2t^2\} \\ T &= \{(x_0, y_0, z_0) \mid x_0^2 + y_0^2 + z_0^2 < \rho^2\}. \end{aligned}$$

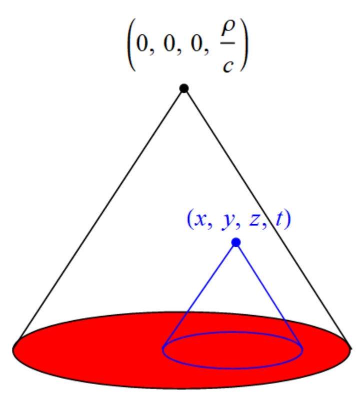
Basically, this surface integral is over the part of the sphere centered at (x, y, z) with radius ct that lies within the solid ball centered at the origin with radius ρ. Depending what region in space-time the point (x, y, z, t) is chosen, the surface integral will yield a different result.

The characteristic surfaces, which are obtained by drawing light rays (with slope c) from every point on the boundary of the hyperdisk within which the initial condition is nonzero, separate these regions.



The red hyperdisk represents the solid ball in $x_0y_0z_0$-space where the initial condition is nonzero. Because it has radius ρ, the height of the cone formed by the crossing characteristic lines is ρ/c. u will be calculated within this cone first.

<u>The First Region</u>

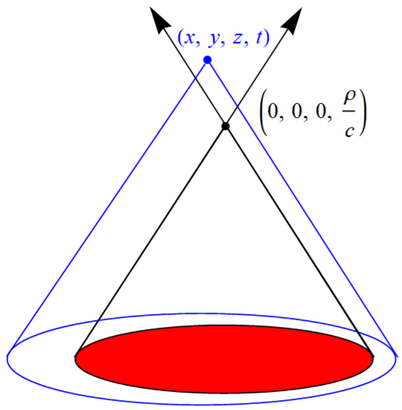


The blue circle illustrated above represents the sphere centered at (x, y, z) with radius ct. Since it lies within the red disk, the surface area of the blue sphere that lies within the red ball is $4\pi(ct)^2$. Consequently,

$$\begin{aligned} u(x, y, z, t) &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint_{Q\cap T} A\, dS_0\right) \\ &= \frac{\partial}{\partial t}\left[\frac{A}{4\pi c^2 t}(4\pi c^2 t^2)\right] \\ &= \frac{\partial}{\partial t}(At) \\ &= A. \end{aligned}$$

This formula for u is valid at points in space-time where $\rho > ct + r$, or $r < \rho - ct$. u will now be calculated in the region directly above the cone.

The Second Region

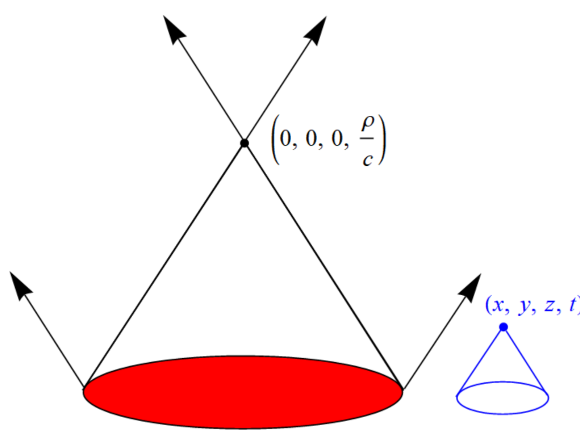


The blue circle illustrated above represents the sphere centered at (x, y, z) with radius ct. Since the red disk lies within it, the surface area of the blue sphere that lies within the red ball is zero. Consequently,

$$\begin{aligned} u(x,y,z,t) &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint_{Q\cap T} A\,dS_0\right) \\ &= \frac{\partial}{\partial t}\left[\frac{A}{4\pi c^2 t}(0)\right] \\ &= 0. \end{aligned}$$

This formula for u is valid at points in space-time where $ct > \rho + r$, or $r < ct - \rho$. u will now be calculated in the region outside the cone right above the $x_0y_0z_0$-plane.

The Third Region

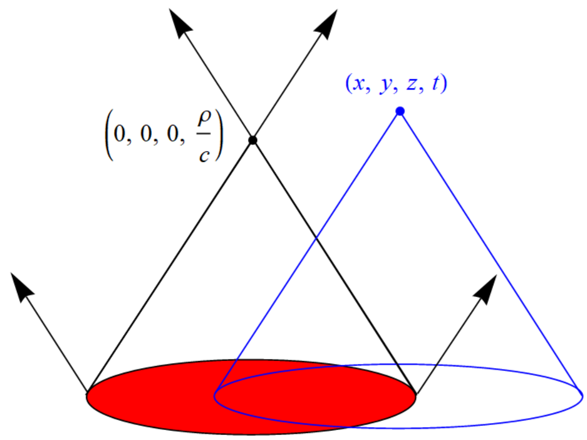


The blue circle illustrated above represents the sphere centered at (x, y, z) with radius ct. Since it and the red disk are completely separate, the surface area of the blue sphere that lies within the red ball is zero. Consequently,

$$\begin{aligned} u(x,y,z,t) &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint_{Q\cap T} A\, dS_0\right) \\ &= \frac{\partial}{\partial t}\left[\frac{A}{4\pi c^2 t}(0)\right] \\ &= 0. \end{aligned}$$

This formula for u is valid at points in space-time where $r > \rho + ct$. u will now be calculated in the last region outside the cone.

<u>The Fourth Region</u>



The blue circle illustrated above represents the sphere centered at (x, y, z) with radius ct. Since it partially intersects the red disk, the surface area of the portion of the blue sphere that lies within the red ball is

$$\frac{\pi(ct)}{r}[\rho^2 - (r-ct)^2].$$

Consequently,

$$\begin{aligned} u(x,y,z,t) &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint_{Q\cap T} A\, dS_0\right) \\ &= \frac{\partial}{\partial t}\left\{\frac{A}{4\pi c^2 t}\frac{\pi(ct)}{r}[\rho^2 - (r-ct)^2]\right\} \\ &= \frac{\partial}{\partial t}\left\{\frac{A}{4cr}[\rho^2 - (r-ct)^2]\right\} \\ &= \frac{A}{4cr}[-2(r-ct)(-c)] \\ &= \frac{A}{2r}(r-ct). \end{aligned}$$

This formula for u is valid at points in space-time where $|\rho - ct| < r < \rho + ct$.

To summarize the results, we have

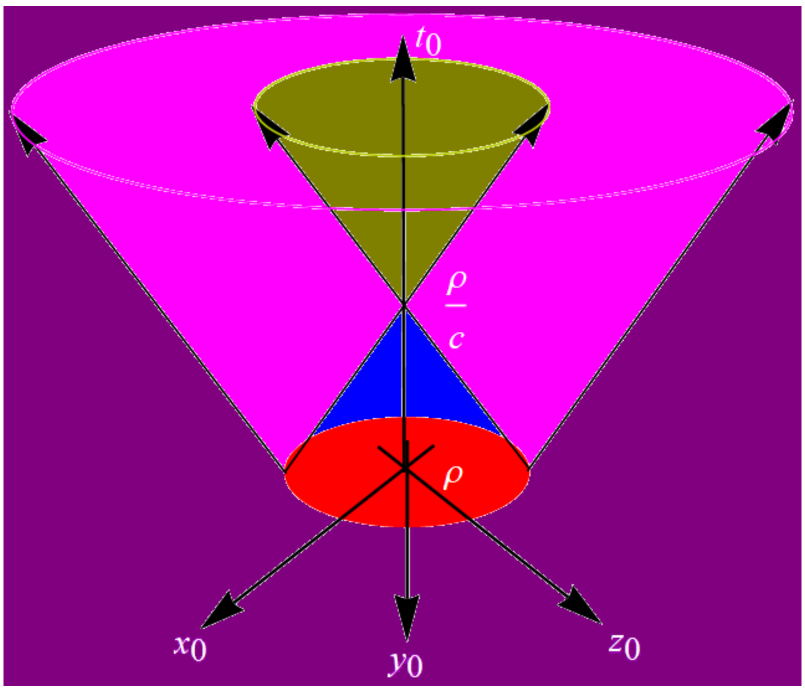
$$u(x,y,z,t) = \begin{cases} A & \text{if } r < \rho - ct \\ 0 & \text{if } r < ct - \rho \\ 0 & \text{if } r > \rho + ct \\ \dfrac{A}{2r}(r - ct) & \text{if } |\rho - ct| < r < \rho + ct \end{cases}.$$

Therefore, replacing r with $\sqrt{x^2+y^2+z^2}$,

$$u(x,y,z,t) = \begin{cases} A & \text{if } \sqrt{x^2+y^2+z^2} < \rho - ct \\ 0 & \text{if } \sqrt{x^2+y^2+z^2} < ct - \rho \\ 0 & \text{if } \sqrt{x^2+y^2+z^2} > \rho + ct \\ \dfrac{A}{2\sqrt{x^2+y^2+z^2}}(\sqrt{x^2+y^2+z^2} - ct) & \text{if } |\rho - ct| < \sqrt{x^2+y^2+z^2} < \rho + ct \end{cases}.$$

Part (b)

Space-time is illustrated below; the solution to the wave equation in each region is labeled by color.



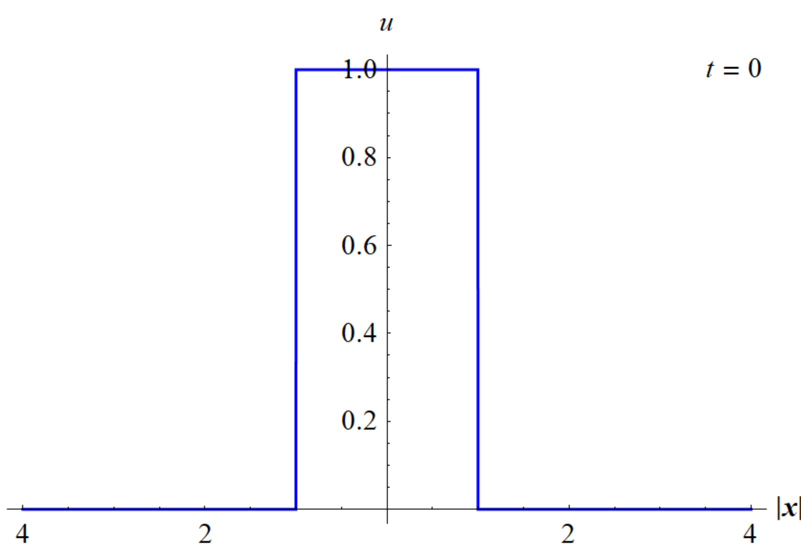
Note that substituting $\rho - ct$, $ct - \rho$, and $\rho + ct$ for $\sqrt{x^2+y^2+z^2}$ in the magenta solution does not result in the blue, olive, and purple solutions, respectively. In other words, u is discontinuous across each region.

A movie of the solution will now be produced. Set $\rho = c = A = 1$ and replace $\sqrt{x^2+y^2+z^2}$ with $|\mathbf{x}|$ in the solution.

$$u(x,y,z,t) = \begin{cases} 1 & \text{if } |\mathbf{x}| < 1-t \\ 0 & \text{if } |\mathbf{x}| < t-1 \\ 0 & \text{if } |\mathbf{x}| > 1+t \\ \dfrac{1}{2|\mathbf{x}|}(|\mathbf{x}|-t) & \text{if } |1-t| < |\mathbf{x}| < 1+t \end{cases}$$

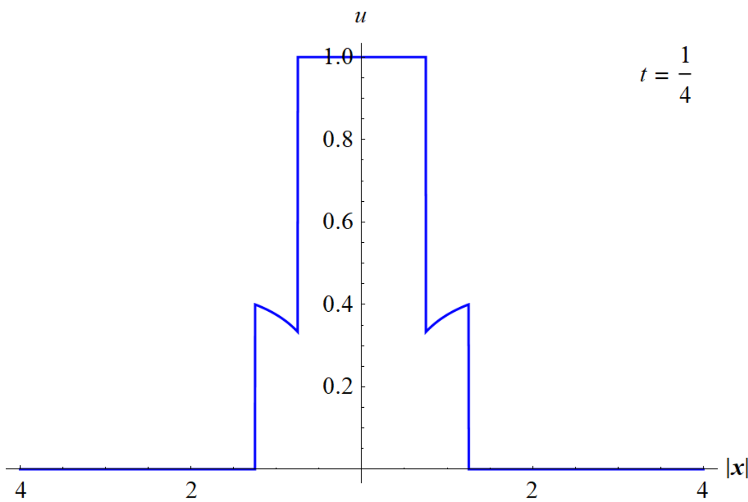
u is only a function of $|\mathbf{x}|$ and t, so $u = u(|\mathbf{x}|, t)$. If $t = 0$, then

$$u(|\mathbf{x}|, 0) = \begin{cases} 1 & \text{if } |\mathbf{x}| < 1 \\ 0 & \text{if } |\mathbf{x}| < -1 \\ 0 & \text{if } |\mathbf{x}| > 1 \\ \dfrac{1}{2} & \text{if } 1 < |\mathbf{x}| < 1 \end{cases}.$$



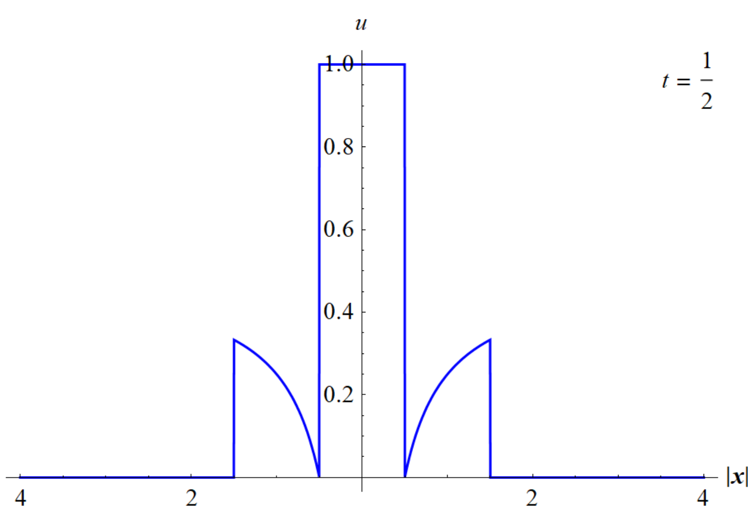
If $t = \frac{1}{4}$, then

$$u\left(|\mathbf{x}|, \frac{1}{4}\right) = \begin{cases} 1 & \text{if } |\mathbf{x}| < \frac{3}{4} \\ 0 & \text{if } |\mathbf{x}| < -\frac{3}{4} \\ 0 & \text{if } |\mathbf{x}| > \frac{5}{4} \\ \frac{1}{2} - \frac{1}{8|\mathbf{x}|} & \text{if } \frac{3}{4} < |\mathbf{x}| < \frac{5}{4} \end{cases}.$$



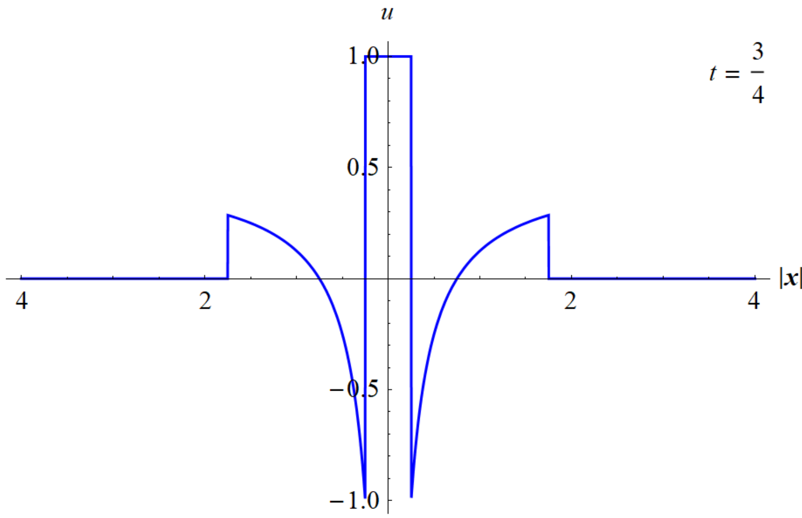
If $t = \frac{1}{2}$, then

$$u\left(|\mathbf{x}|, \frac{1}{2}\right) = \begin{cases} 1 & \text{if } |\mathbf{x}| < \frac{1}{2} \\ 0 & \text{if } |\mathbf{x}| < -\frac{1}{2} \\ 0 & \text{if } |\mathbf{x}| > \frac{3}{2} \\ \frac{1}{2} - \frac{1}{4|\mathbf{x}|} & \text{if } \frac{1}{2} < |\mathbf{x}| < \frac{3}{2} \end{cases}.$$



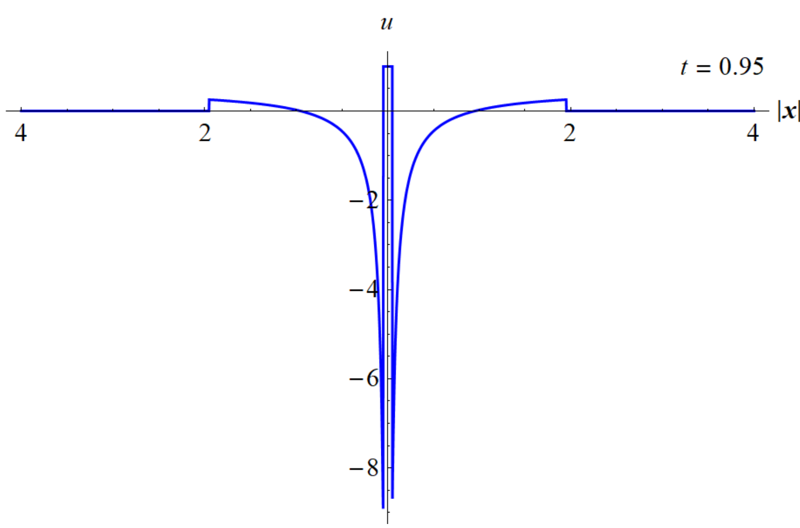
If $t = \frac{3}{4}$, then

$$u\left(|\mathbf{x}|, \frac{3}{4}\right) = \begin{cases} 1 & \text{if } |\mathbf{x}| < \frac{1}{4} \\ 0 & \text{if } |\mathbf{x}| < -\frac{1}{4} \\ 0 & \text{if } |\mathbf{x}| > \frac{7}{4} \\ \frac{1}{2} - \frac{3}{8|\mathbf{x}|} & \text{if } \frac{1}{4} < |\mathbf{x}| < \frac{7}{4} \end{cases}.$$



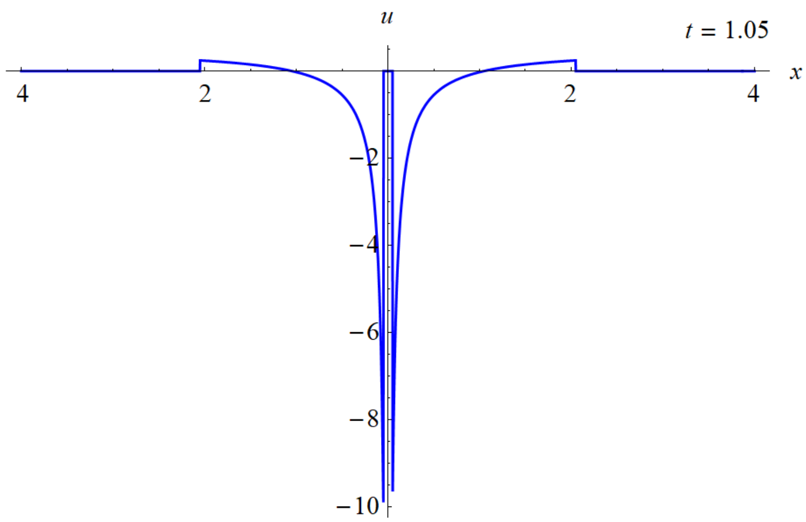
If $t = 0.95$, then

$$u(|\mathbf{x}|, 0.95) = \begin{cases} 1 & \text{if } |\mathbf{x}| < 0.05 \\ 0 & \text{if } |\mathbf{x}| < -0.05 \\ 0 & \text{if } |\mathbf{x}| > 1.95 \\ \frac{1}{2} - \frac{0.95}{2|\mathbf{x}|} & \text{if } 0.05 < |\mathbf{x}| < 1.95 \end{cases}.$$



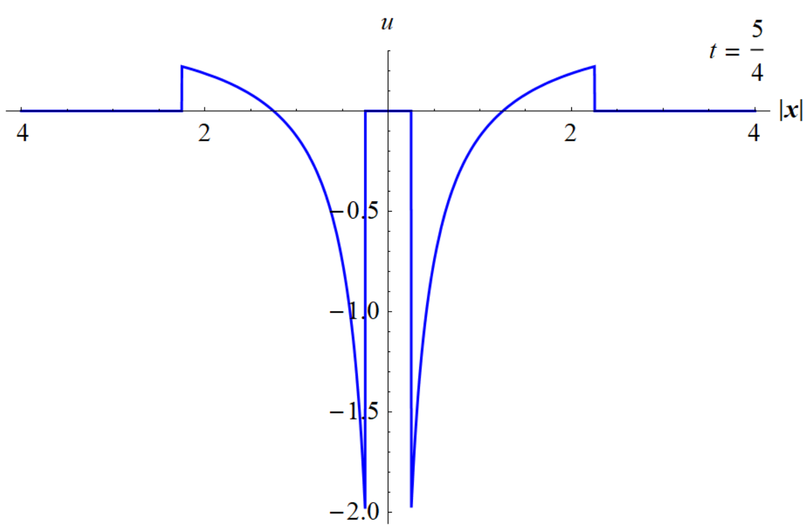
If $t = 1.05$, then

$$u(|\mathbf{x}|, 1.05) = \begin{cases} 1 & \text{if } |\mathbf{x}| < -0.05 \\ 0 & \text{if } |\mathbf{x}| < 0.05 \\ 0 & \text{if } |\mathbf{x}| > 2.05 \\ \frac{1}{2} - \frac{1.05}{2|\mathbf{x}|} & \text{if } 0.05 < |\mathbf{x}| < 2.05 \end{cases}.$$



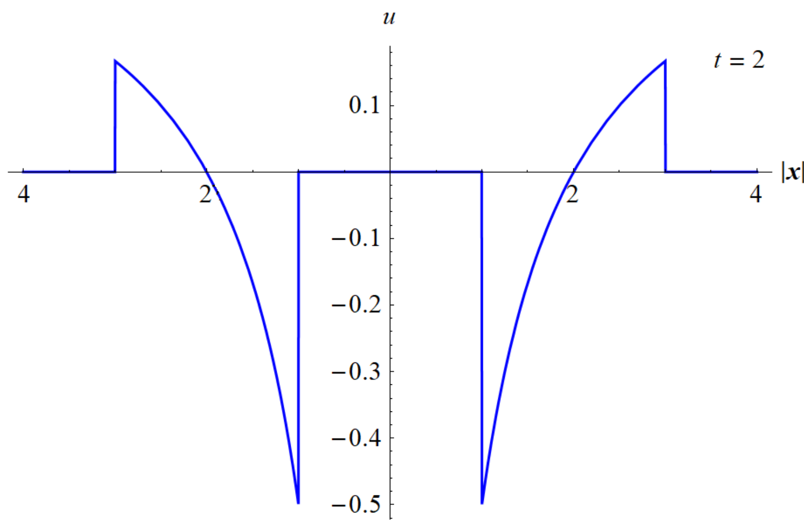
If $t = \frac{5}{4}$, then

$$u\left(|\mathbf{x}|, \frac{5}{4}\right) = \begin{cases} 1 & \text{if } |\mathbf{x}| < -\frac{1}{4} \\ 0 & \text{if } |\mathbf{x}| < \frac{1}{4} \\ 0 & \text{if } |\mathbf{x}| > \frac{9}{4} \\ \frac{1}{2} - \frac{5}{8|\mathbf{x}|} & \text{if } \frac{1}{4} < |\mathbf{x}| < \frac{9}{4} \end{cases}.$$



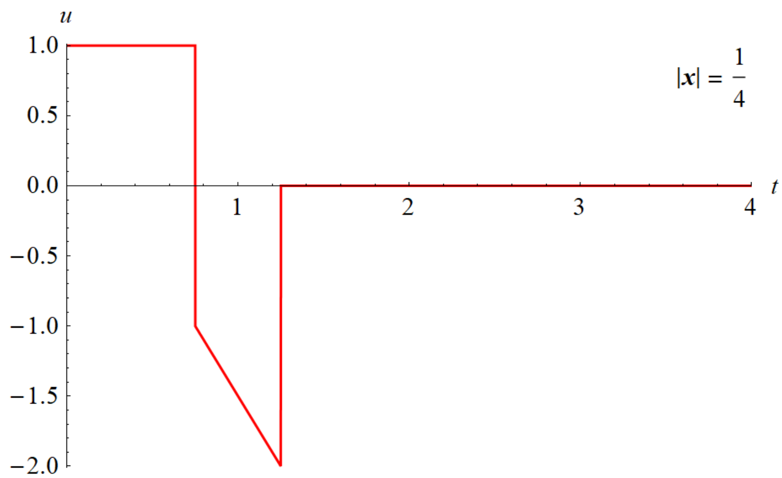
If $t = 2$, then

$$u(|\mathbf{x}|, 2) = \begin{cases} 1 & \text{if } |\mathbf{x}| < -1 \\ 0 & \text{if } |\mathbf{x}| < 1 \\ 0 & \text{if } |\mathbf{x}| > 3 \\ \dfrac{1}{2} - \dfrac{1}{|\mathbf{x}|} & \text{if } 1 < |\mathbf{x}| < 3 \end{cases}.$$



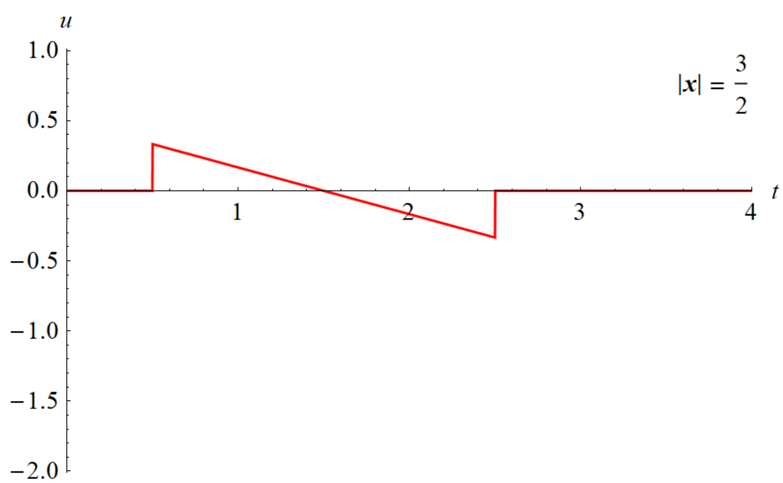
Plots of the amplitude will be shown for observers standing at $|\mathbf{x}| = \frac{1}{4}$ and $|\mathbf{x}| = \frac{3}{2}$. If $|\mathbf{x}| = \frac{1}{4}$, then

$$u\left(\frac{1}{4}, t\right) = \begin{cases} 1 & \text{if } t < \frac{3}{4} \\ 0 & \text{if } t > \frac{5}{4} \\ 0 & \text{if } t < -\frac{3}{4} \\ \frac{1}{2} - 2t & \text{if } \frac{3}{4} < t < \frac{5}{4} \end{cases}.$$



If $|\mathbf{x}| = \frac{3}{2}$, then

$$u\left(\frac{3}{2}, t\right) = \begin{cases} 1 & \text{if } t < -\frac{1}{2} \\ 0 & \text{if } t > \frac{5}{2} \\ 0 & \text{if } t < \frac{1}{2} \\ \frac{1}{2} - \frac{t}{3} & \text{if } \frac{1}{2} < t < \frac{5}{2} \end{cases}.$$



Part (c)

Let $\mathbf{x}_1$ be a point in space-time within the red hyperdisk initially: $|\mathbf{x}_1| < \rho$ when $t = 0$. The aim here is to show that

$$\lim_{t\to\infty} t \cdot u(\mathbf{x}_1 + t\mathbf{v}, t)$$

converges, where $|\mathbf{v}| = c$. Replace $\sqrt{x^2+y^2+z^2}$ with $|\mathbf{x}|$ and (x, y, z) with $\mathbf{x}$ in the solution for u.

$$u(\mathbf{x}, t) = \begin{cases} A & \text{if } |\mathbf{x}| < \rho - ct \\ 0 & \text{if } |\mathbf{x}| < ct - \rho \\ 0 & \text{if } |\mathbf{x}| > \rho + ct \\ \dfrac{A}{2|\mathbf{x}|}(|\mathbf{x}| - ct) & \text{if } |\rho - ct| < |\mathbf{x}| < \rho + ct \end{cases}$$

$(\mathbf{x}_1 + t\mathbf{v}, t)$ lies within the magenta region for large t, so

$$\begin{aligned}
\lim_{t\to\infty} t \cdot u(\mathbf{x}_1 + t\mathbf{v}, t) &= \lim_{t\to\infty} t \cdot \frac{A}{2|\mathbf{x}_1 + t\mathbf{v}|}(|\mathbf{x}_1 + t\mathbf{v}| - ct) \\
&= \lim_{t\to\infty} \frac{At}{2\sqrt{(\mathbf{x}_1 + t\mathbf{v})^2}}\left(\sqrt{(\mathbf{x}_1 + t\mathbf{v})^2} - ct\right) \\
&= \lim_{t\to\infty} \frac{At}{2\sqrt{|\mathbf{x}_1|^2 + 2t\mathbf{x}_1\cdot\mathbf{v} + t^2|\mathbf{v}|^2}}\left(\sqrt{|\mathbf{x}_1|^2 + 2t\mathbf{x}_1\cdot\mathbf{v} + t^2|\mathbf{v}|^2} - ct\right) \\
&= \lim_{t\to\infty} \frac{At}{2\sqrt{|\mathbf{x}_1|^2 + 2t\mathbf{x}_1\cdot\mathbf{v} + c^2t^2}}\left(\sqrt{|\mathbf{x}_1|^2 + 2t\mathbf{x}_1\cdot\mathbf{v} + c^2t^2} - ct\right) \\
&= \lim_{t\to\infty} \frac{A}{\frac{2}{t}\sqrt{|\mathbf{x}_1|^2 + 2t\mathbf{x}_1\cdot\mathbf{v} + c^2t^2}} \cdot ct\left(\frac{1}{ct}\sqrt{|\mathbf{x}_1|^2 + 2t\mathbf{x}_1\cdot\mathbf{v} + c^2t^2} - 1\right) \\
&= \lim_{t\to\infty} \frac{A}{2\sqrt{\frac{|\mathbf{x}_1|^2}{t^2} + \frac{2\mathbf{x}_1\cdot\mathbf{v}}{t} + c^2}} \cdot ct\left(\sqrt{\frac{|\mathbf{x}_1|^2}{c^2t^2} + \frac{2\mathbf{x}_1\cdot\mathbf{v}}{c^2t} + 1} - 1\right) \\
&= \left(\lim_{t\to\infty} \frac{A}{2\sqrt{\cancelto{0}{\frac{|\mathbf{x}_1|^2}{t^2}} + \cancelto{0}{\frac{2\mathbf{x}_1\cdot\mathbf{v}}{t}} + c^2}}\right) \lim_{t\to\infty} ct\left(\sqrt{1 + \frac{2\mathbf{x}_1\cdot\mathbf{v}}{c^2t} + \frac{|\mathbf{x}_1|^2}{c^2t^2}} - 1\right).
\end{aligned}$$

Use the binomial series for the square root in the second limit.

$$\begin{aligned}
&= \left(\frac{A}{2c}\right) \lim_{t\to\infty} ct\left[1 + \frac{\mathbf{x}_1\cdot\mathbf{v}}{c^2t} + \frac{|\mathbf{x}_1|^2}{2c^2t^2} + O\left(\frac{1}{t^2}\right) - 1\right] \\
&= \frac{A}{2c}\lim_{t\to\infty}\left[\frac{\mathbf{x}_1\cdot\mathbf{v}}{c} + \cancelto{0}{\frac{|\mathbf{x}_1|^2}{2ct}} + \cancelto{0}{O\left(\frac{1}{t}\right)}\right] \\
&= \frac{A}{2c^2}\mathbf{x}_1\cdot\mathbf{v}
\end{aligned}$$

Therefore, $t \cdot u(\mathbf{x}_1 + t\mathbf{v}, t)$ converges as $t \to \infty$.