Exercise 9

(a) For any solution of the three-dimensional wave equation with initial data vanishing outside some sphere, show that $u(x, y, z, t) = 0$ for fixed (x, y, z) and large enough t.

(b) Prove that $u(x, y, z, t) = O(t^{-1})$ *uniformly* as $t \to \infty$; that is, prove that $t \cdot u(x, y, z, t)$ is a bounded function of x, y, z, and t. (*Hint:* Use Kirchhoff's formula.)

Solution

Part (a)

Suppose that the initial conditions for the three-dimensional wave equation are nonzero in a solid ball centered at (x_1, y_1, z_1) with radius ρ.

$$u_{tt} = c^2\nabla^2 u, \quad -\infty < x, y, z < \infty,\ t > 0$$
$$u(x, y, z, 0) = \begin{cases} \alpha(x, y, z) & (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \le \rho^2 \\ 0 & (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 > \rho^2 \end{cases}$$
$$u_t(x, y, z, 0) = \begin{cases} \beta(x, y, z) & (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \le \rho^2 \\ 0 & (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 > \rho^2 \end{cases}$$

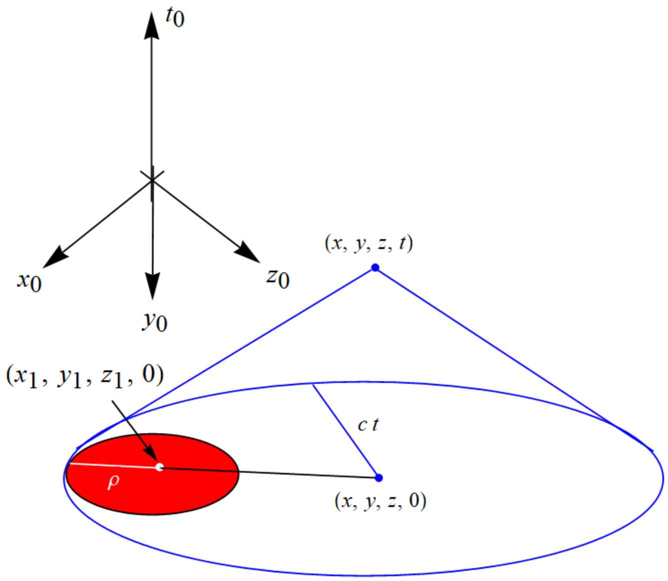
The solution of this initial value problem is given by the formula of Kirchhoff and Poisson,

$$u(x, y, z, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint\limits_{Q\cap T} \alpha(x_0, y_0, z_0)\, dS_0\right] + \frac{1}{4\pi c^2 t}\iint\limits_{Q\cap T} \beta(x_0, y_0, z_0)\, dS_0,$$

where

$$Q = \{(x_0, y_0, z_0) \mid (x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 = c^2t^2\}$$
$$T = \{(x_0, y_0, z_0) \mid (x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2 \le \rho^2\}.$$

Basically, these surface integrals are over the portion of the sphere centered at (x, y, z) with radius ct that lies within the solid ball centered at (x_1, y_1, z_1) with radius ρ. This sphere and solid ball are represented by the blue hypercircle and the red hyperdisk, respectively, in the $x_0y_0z_0$-plane below.



If

$$ct > \rho + \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2},$$

then the intersection of Q and T is the empty set. Therefore, $u(x, y, z, t) = 0$ if

$$t > \frac{\rho + \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2}}{c}.$$

Part (b)

Assuming that $\alpha(x, y, z)$ and $\beta(x, y, z)$ are continuous functions, let A and B be the respective maxima of these functions over the solid ball where u is nonzero initially. The solution is then

$$\begin{aligned}
u(x, y, z, t) &= \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint_{Q\cap T} \alpha(x_0, y_0, z_0)\, dS_0\right] + \frac{1}{4\pi c^2 t}\iint_{Q\cap T} \beta(x_0, y_0, z_0)\, dS_0 \\
&\le \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint_{Q\cap T} A\, dS_0\right] + \frac{1}{4\pi c^2 t}\iint_{Q\cap T} B\, dS_0 \\
&= \frac{\partial}{\partial t}\left[\frac{A}{4\pi c^2 t}\iint_{Q\cap T} dS_0\right] + \frac{B}{4\pi c^2 t}\iint_{Q\cap T} dS_0.
\end{aligned}$$

The maximum surface area that can be enclosed in the solid ball is $4\pi\rho^2$.

$$\begin{aligned}
&\le \frac{\partial}{\partial t}\left[\frac{A}{4\pi c^2 t}\cdot(4\pi\rho^2)\right] + \frac{B}{4\pi c^2 t}\cdot(4\pi\rho^2) \\
&= \frac{\partial}{\partial t}\left(\frac{A\rho^2}{c^2 t}\right) + \frac{B\rho^2}{c^2 t} \\
&= -\frac{A\rho^2}{c^2 t^2} + \frac{B\rho^2}{c^2 t}
\end{aligned}$$

Multiply both sides by t.

$$t \cdot u(x, y, z, t) \leq -\frac{A\rho^2}{c^2 t} + \frac{B\rho^2}{c^2}$$

Take the limit of both sides as $t \to \infty$.

$$\lim_{t \to \infty} t \cdot u(x, y, z, t) \leq \frac{B\rho^2}{c^2}.$$

Therefore, since $t \cdot u(x, y, z, t)$ is a bounded function as $t \to \infty$, $u(x, y, z, t) = O(t^{-1})$ uniformly as $t \to \infty$.