Exercise 12

Solve the three-dimensional wave equation in $\{r \neq 0,\ t > 0\}$ with zero initial conditions and with the limiting condition

$$\lim_{r\to 0} 4\pi r^2 u_r(r,t) = g(t).$$

Assume that $g(0) = g'(0) = g''(0) = 0$.

Solution

The initial boundary value problem we wish to solve is as follows.

$$\begin{gathered} u_{tt} = c^2\nabla^2 u, \quad -\infty < x, y, z < \infty,\ t > 0 \\ u(x,y,z,0) = 0 \qquad u_t(x,y,z,0) = 0 \\ \lim_{r\to 0} 4\pi r^2 u_r(r,t) = g(t) \end{gathered}$$

u is assumed to be spherically symmetric based on the form of the boundary condition and zero initial conditions. As a result, all angular derivatives in the Laplacian operator vanish when it's expanded in spherical coordinates (r, ϕ, θ), where θ represents the angle from the polar axis.

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r + \underbrace{\frac{1}{r^2}u_{\theta\theta}}_{=0} + \underbrace{\frac{\cot\theta}{r^2}u_\theta}_{=0} + \underbrace{\frac{1}{r^2\sin^2\theta}u_{\phi\phi}}_{=0}\right)$$

The problem has been reduced to the spherical wave equation on a semi-infinite interval.

$$\begin{gathered} u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right), \quad 0 < r < \infty,\ t > 0 \\ u(r,0) = 0 \qquad u_t(r,0) = 0 \\ \lim_{r\to 0} 4\pi r^2 u_r(r,t) = g(t) \end{gathered}$$

As in Exercise 2.1.8, make the substitution $v(r,t) = ru(r,t)$ in order to transform this PDE to the wave equation. Find the derivatives of u in terms of this new variable.

$$\begin{aligned} v_t &= ru_t \\ v_{tt} &= ru_{tt} \quad \rightarrow \quad u_{tt} = \frac{v_{tt}}{r} \\ v_r &= u + ru_r \\ v_{rr} &= u_r + u_r + ru_{rr} = 2u_r + ru_{rr} \quad \rightarrow \quad u_{rr} + \frac{2}{r}u_r = \frac{v_{rr}}{r} \end{aligned}$$

v satisfies the one-dimensional wave equation.

$$\frac{v_{tt}}{r} = c^2\left(\frac{v_{rr}}{r}\right)$$

$$v_{tt} = c^2 v_{rr}$$

The initial conditions for v are

$$\begin{gathered} v(r,0) = ru(r,0) = 0 \\ v_t(r,0) = ru_t(r,0) = 0, \end{gathered}$$

and the boundary condition for v is

$$\lim_{r\to 0} 4\pi r^2 \frac{\partial}{\partial r}\left[\frac{v(r,t)}{r}\right] = g(t)$$

$$\lim_{r\to 0} 4\pi r^2 \left[\frac{r v_r(r,t) - v(r,t)}{r^2}\right] = g(t)$$

$$4\pi \lim_{r\to 0}[r v_r(r,t) - v(r,t)] = g(t)$$

$$4\pi[-v(0,t)] = g(t)$$

$$v(0,t) = -\frac{g(t)}{4\pi}.$$

To summarize, u will be determined by solving the much simpler initial boundary value problem for v.

$$v_{tt} - c^2 v_{rr} = 0, \quad 0 < r < \infty,\ t > 0$$
$$v(r,0) = 0 \qquad v_t(r,0) = 0$$
$$v(0,t) = -\frac{g(t)}{4\pi}$$

Comparing the wave equation to the general form of a second-order PDE,

$$Au_{tt} + Bu_{rt} + Cu_{rr} + Du_t + Eu_r + Fu = G,$$

we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations for a second-order PDE are given by

$$\begin{aligned}\frac{dr}{dt} &= \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC}) \\ &= \frac{1}{2}(\pm\sqrt{4c^2}) \\ &= \pm c.\end{aligned}$$

Because the discriminant $B^2 - 4AC = 4c^2$ is positive, the two families of characteristic curves are real and distinct. In particular, they are lines with slopes $\pm c$ and characteristic coordinates, ξ and η, respectively.

$$\frac{dr}{dt} = c \quad \rightarrow \quad r = ct + \xi$$
$$\frac{dr}{dt} = -c \quad \rightarrow \quad r = -ct + \eta$$

Suppose we are interested in evaluating v at the point (r_0, t_0). The equations of the lines going through this point are

$$r - r_0 = c(t - t_0)$$
$$r - r_0 = -c(t - t_0).$$

As shown in the figure below, if (r_0, t_0) lies in the domain $r + ct > 0$, then the solution behaves as if there were no boundary. On the other hand, if (r_0, t_0) lies in the domain $r - ct < 0$, then a reflection occurs at the boundary. The solution has to be considered in each case.

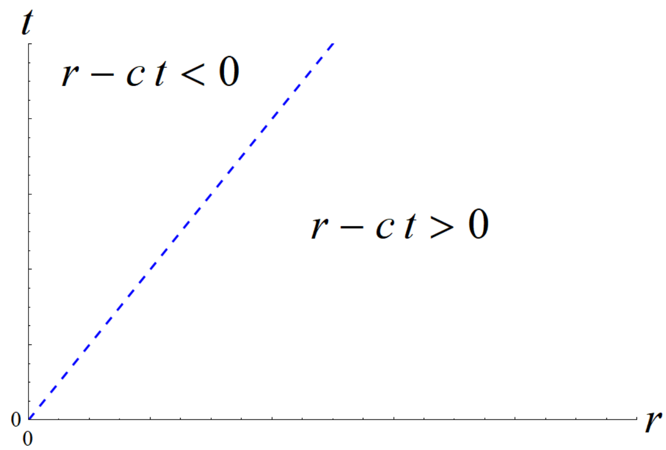
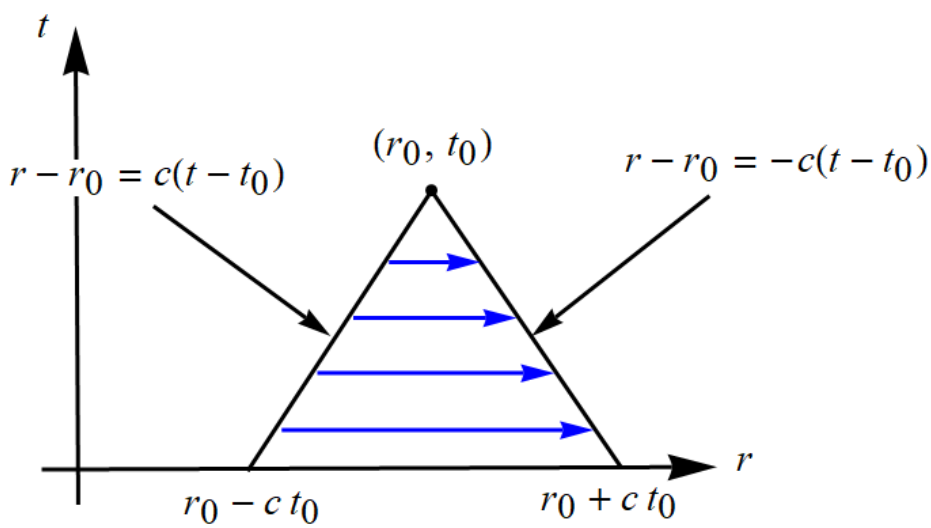


Figure 1: The presence of a boundary at $r = 0$ means we have to consider the solution to the PDE in the domains above and below the line $r - ct = 0$. The reason is that a reflection occurs for points above it but not below it.

Case 1: $r - ct > 0$



No reflection occurs in this case. Integrate both sides of the PDE over the triangular domain D_1 enclosed by the lines (from left to right as indicated above).

$$\iint_{D_1} (v_{tt} - c^2 v_{rr})\, dA = 0$$

Rewrite the left side.

$$-\iint_{D_1} \left[\frac{\partial}{\partial r}(c^2 v_r) - \frac{\partial}{\partial t}(v_t)\right] dA = 0$$

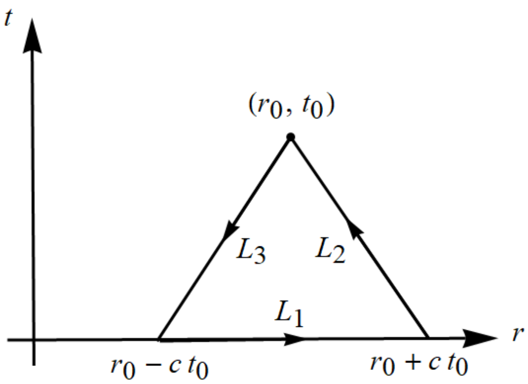
Multiply both sides by -1.

$$\iint\limits_{D_1} \left[\frac{\partial}{\partial r}(c^2 v_r) - \frac{\partial}{\partial t}(v_t)\right] dA = 0$$

Apply Green's theorem (essentially the divergence theorem in two dimensions) to the double integral to turn it into a counterclockwise line integral around the triangle's boundary bdy D_1.

$$\oint\limits_{\text{bdy } D_1} (v_t\, dx + c^2 v_r\, dt) = 0$$

Let L_1, L_2, and L_3 represent the legs of the triangle.



The line integral is the sum of three integrals, one over each leg.

$$\int_{L_1} (v_t\, dr + c^2 v_r\, dt) + \int_{L_2} (v_t\, dr + c^2 v_r\, dt) + \int_{L_3} (v_t\, dr + c^2 v_r\, dt) = 0$$

On L_1	On L_2	On L_3
$t = 0$	$r - r_0 = -c(t - t_0)$	$r - r_0 = c(t - t_0)$
$dt = 0$	$dr = -c\, dt$	$dr = c\, dt$

Replace the differentials in the integrals over L_2 and L_3.

$$\int_{r_0 - ct_0}^{r_0 + ct_0} v_t(r, 0)\, dr + \int_{L_2} \left[v_t(-c\, dt) + c^2 v_r \left(-\frac{dr}{c}\right)\right] + \int_{L_3} \left[v_t(c\, dt) + c^2 v_r \left(\frac{dr}{c}\right)\right] = 0$$

In this exercise $v_t(r, 0) = 0$, so the integral over L_1 vanishes.

$$-c \int_{L_2} \left(\frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial r} dr\right) + c \int_{L_3} \left(\frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial r} dr\right) = 0$$

The remaining integrands are how the differential of $v = v(r, t)$ is defined.

$$-c \int_{L_2} dv + c \int_{L_3} dv = 0$$

Evaluate the remaining integrals.

$$-c[v(r_0, t_0) - v(r_0 + ct_0, 0)] + c[v(r_0 - ct_0, 0) - v(r_0, t_0)] = 0$$

In this exercise $v(r,0) = 0$, so $v(r_0 + ct_0, 0) = 0$ and $v(r_0 - ct_0, 0) = 0$.

$$-2cv(r_0, t_0) = 0$$

Divide both sides by $-2c$.

$$v(r_0, t_0) = 0$$

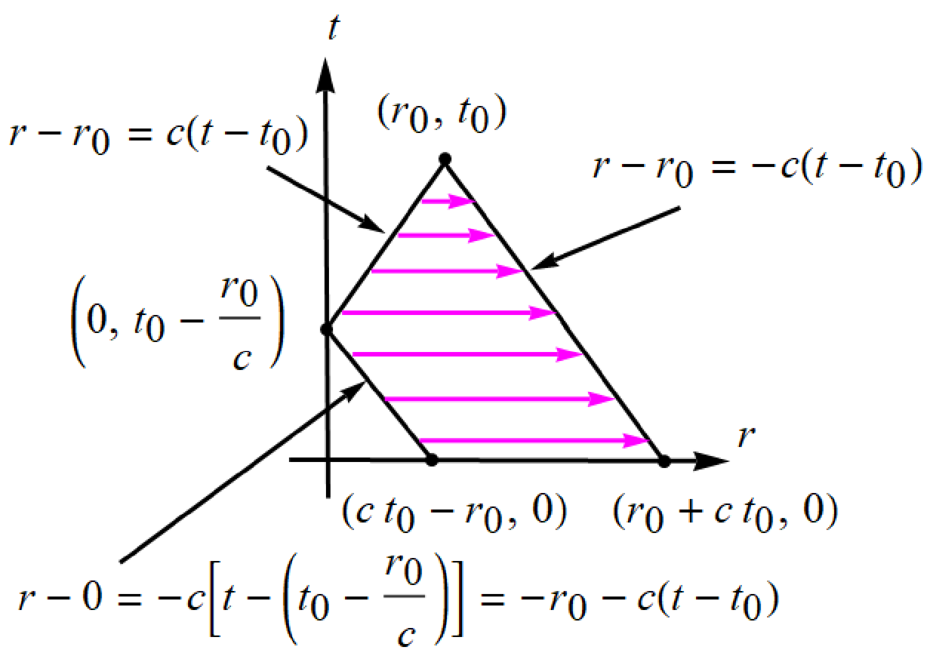
Switch the roles of r and t with those of r_0 and t_0, respectively.

$$v(r,t) = 0, \quad r - ct > 0$$

Therefore, since $u(r,t) = v(r,t)/r$,

$$u(r,t) = 0, \quad r - ct > 0.$$

Case 2: $r - ct < 0$



Integrate both sides of the PDE over the polygonal domain D_2 enclosed by the lines (from left to right as indicated above).

$$\iint_{D_2} (v_{tt} - c^2 v_{rr})\, dA = 0$$

Rewrite the left side.

$$-\iint_{D_2} \left[\frac{\partial}{\partial r}(c^2 v_r) - \frac{\partial}{\partial t}(v_t)\right] dA = 0$$

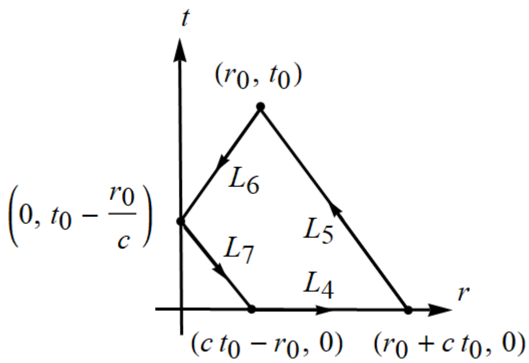
Multiply both sides by -1.

$$\iint_{D_2} \left[\frac{\partial}{\partial r}(c^2 v_r) - \frac{\partial}{\partial t}(v_t)\right] dA = 0$$

Apply Green's theorem to the double integral to turn it into a counterclockwise line integral around the polygon's boundary bdy D_2.

$$\oint_{\text{bdy } D_2} (v_t\, dr + c^2 v_r\, dt) = 0$$

Let L_4, L_5, L_6, and L_7 represent the legs of the polygon.



The line integral is the sum of four integrals, one over each leg.

$$\int_{L_4} (v_t\, dr + c^2 v_r\, dt) + \int_{L_5} (v_t\, dr + c^2 v_r\, dt) + \int_{L_6} (v_t\, dr + c^2 v_r\, dt) + \int_{L_7} (v_t\, dr + c^2 v_r\, dt) = 0$$

On L_4	On L_5	On L_6	On L_7
$t = 0$	$r - r_0 = -c(t - t_0)$	$r - r_0 = c(t - t_0)$	$r = -r_0 - c(t - t_0)$
$dt = 0$	$dr = -c\, dt$	$dr = c\, dt$	$dr = -c\, dt$

Replace the differentials in the integrals over L_5, L_6, and L_7.

$$\int_{ct_0 - r_0}^{r_0 + ct_0} v_t(r, 0)\, dr + \int_{L_5} \left[v_t(-c\, dt) + c^2 v_r \left(-\frac{dr}{c} \right) \right] + \int_{L_6} \left[v_t(c\, dt) + c^2 v_r \left(\frac{dr}{c} \right) \right] + \int_{L_7} \left[v_t(-c\, dt) + c^2 v_r \left(-\frac{dr}{c} \right) \right] = 0$$

In this exercise $v_t(r, 0) = 0$, so the integral over L_4 vanishes.

$$-c \int_{L_5} \left(\frac{\partial v}{\partial t}\, dt + \frac{\partial v}{\partial r}\, dr \right) + c \int_{L_6} \left(\frac{\partial v}{\partial t}\, dt + \frac{\partial v}{\partial r}\, dr \right) - c \int_{L_7} \left(\frac{\partial v}{\partial t}\, dt + \frac{\partial v}{\partial r}\, dr \right) = 0$$

The remaining integrands on the left side are how the differential of $v = v(r, t)$ is defined.

$$-c \int_{L_5} dv + c \int_{L_6} dv - c \int_{L_7} dv = 0$$

Evaluate the remaining integrals.

$$-c[v(r_0, t_0) - v(r_0 + ct_0, 0)] + c \left[v\left(0, t_0 - \frac{r_0}{c}\right) - v(r_0, t_0) \right] - c \left[v(ct_0 - r_0, 0) - v\left(0, t_0 - \frac{r_0}{c}\right) \right] = 0$$

In this exercise $v(r,0) = 0$ and $v(0,t) = -g(t)/4\pi$, so $v(r_0 + ct_0, 0) = 0$ and $v(ct_0 - r_0, 0) = 0$ and $v(0, t_0 - r_0/c) = -g(t_0 - r_0/c)/4\pi$.

$$-2cv(r_0, t_0) + 2c\left[-\frac{1}{4\pi}g\left(t_0 - \frac{r_0}{c}\right)\right] = 0$$

Solve this equation for $v(r_0, t_0)$.

$$v(r_0, t_0) = -\frac{1}{4\pi}g\left(t_0 - \frac{r_0}{c}\right)$$

Switch the roles of r and t with those of r_0 and t_0, respectively.

$$v(r,t) = -\frac{1}{4\pi}g\left(t - \frac{r}{c}\right), \quad r - ct < 0$$

Therefore, since $u(r,t) = v(r,t)/r$,

$$u(r,t) = -\frac{1}{4\pi r}g\left(t - \frac{r}{c}\right), \quad r - ct < 0.$$

In conclusion, the solution to the initial boundary value problem is

$$u(r,t) = \begin{cases} -\frac{1}{4\pi r}g\left(t - \frac{r}{c}\right) & \text{if } r - ct < 0 \\ 0 & \text{if } r - ct > 0 \end{cases}.$$