Exercise 16

(a) Solve the wave equation in two dimensions for $t > 0$ with the initial conditions $\phi(\mathbf{x}) \equiv 0$, $\psi(\mathbf{x}) = A$ for $|\mathbf{x}| < \rho$, and $\psi(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho$, where A is a constant. Do not carry out the integral.

(b) Under the same conditions find a simple formula for the solution $u(\mathbf{0}, t)$ at the origin by carrying out the integral.

Solution

Part (a)

The solution of the two-dimensional wave equation in space subject to two initial conditions,

$$\begin{aligned} &u_{tt} = c^2 \nabla^2 u, \quad -\infty < x, y < \infty, \ t > 0 \\ &u(x, y, 0) = \alpha(x, y) \\ &u_t(x, y, 0) = \beta(x, y), \end{aligned}$$

is given by

$$\begin{aligned} u(x, y, t) = \frac{\partial}{\partial t}\left[\frac{1}{2\pi c} \iint\limits_{\substack{(x_0 - x)^2 + (y_0 - y)^2 \\ \leq c^2 t^2}} \frac{\alpha(x_0, y_0)}{\sqrt{c^2 t^2 - (x_0 - x)^2 - (y_0 - y)^2}} \, dx_0 \, dy_0\right] \\ + \frac{1}{2\pi c} \iint\limits_{\substack{(x_0 - x)^2 + (y_0 - y)^2 \\ \leq c^2 t^2}} \frac{\beta(x_0, y_0)}{\sqrt{c^2 t^2 - (x_0 - x)^2 - (y_0 - y)^2}} \, dx_0 \, dy_0. \end{aligned}$$

In particular, we wish to solve the initial value problem when

$$\alpha(x, y) = 0 \quad \text{and} \quad \beta(x, y) = \begin{cases} A & r < \rho \\ 0 & r > \rho \end{cases},$$

where $r = |\mathbf{x}| = \sqrt{x^2 + y^2}$. With these initial conditions, the previous formula simplifies to

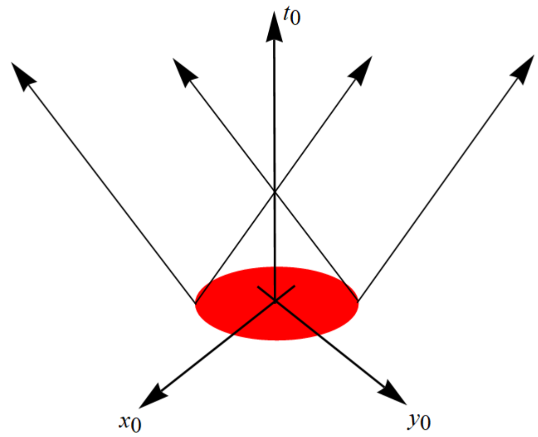
$$u(x, y, t) = \frac{1}{2\pi c} \iint\limits_{Q \cap T} \frac{A}{\sqrt{c^2 t^2 - (x_0 - x)^2 - (y_0 - y)^2}} \, dx_0 \, dy_0,$$

where

$$\begin{aligned} Q &= \{(x_0, y_0) \mid (x_0 - x)^2 + (y_0 - y)^2 \leq c^2 t^2\} \\ T &= \{(x_0, y_0) \mid x_0^2 + y_0^2 < \rho^2\}. \end{aligned}$$

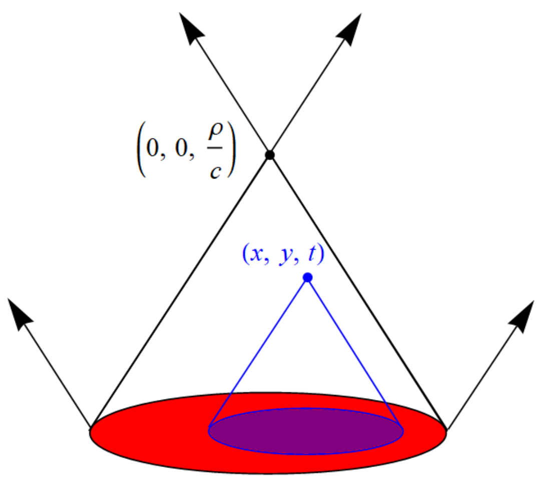
Basically, this double integral is over the area of the disk centered at (x, y) with radius ct that lies within the disk centered at the origin with radius ρ. Depending what region in space-time the point (x, y, t) is chosen, the double integral will yield a different result.

The characteristic surfaces, which are obtained by drawing light rays (with slope c) from every point on the boundary of the disk within which the initial condition is nonzero, separate these regions.



The red disk in the x_0y_0-plane is where the initial condition is nonzero. Because it has radius ρ, the height of the cone formed by the crossing characteristic lines is ρ/c. u will be calculated within this cone first.

The First Region



The purple area illustrated above represents the intersection of the blue disk centered at (x, y) with radius ct with the red disk. Letting $x_0 - x = r_0 \cos\theta_0$ and $y_0 - y = r_0 \sin\theta_0$, the double integral over this purple area is

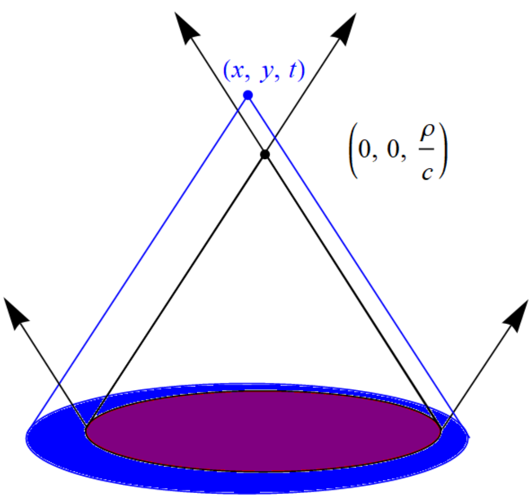
$$\begin{aligned} u(x, y, t) &= \frac{1}{2\pi c} \iint_{Q \cap T} \frac{A}{\sqrt{c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2}}\, dx_0\, dy_0 \\ &= \frac{1}{2\pi c} \int_0^{2\pi} \int_0^{ct} \frac{A}{\sqrt{c^2t^2 - r_0^2 \cos^2\theta_0 - r_0^2 \sin^2\theta_0}} (r_0\, dr_0\, d\theta_0) \\ &= \frac{A}{2\pi c} \int_0^{2\pi} \int_0^{ct} \frac{r_0}{\sqrt{c^2t^2 - r_0^2}}\, dr_0\, d\theta_0. \end{aligned}$$

The integral in dr_0 can be solved with a substitution: $v = c^2t^2 - r_0^2$ and $dv = -2r_0\,dr_0$.

$$\begin{aligned} u(x,y,t) &= \frac{A}{2\pi c}\left(\int_0^{2\pi} d\theta_0\right)\left(\int_0^{ct} \frac{r_0}{\sqrt{c^2t^2 - r_0^2}}\,dr_0\right) \\ &= \frac{A}{2\pi c}(2\pi)(ct) \\ &= At \end{aligned}$$

This formula for u is valid at points in space-time where $\rho > ct + r$, or $r < \rho - ct$. u will now be calculated in the region directly above the cone.

<u>The Second Region</u>

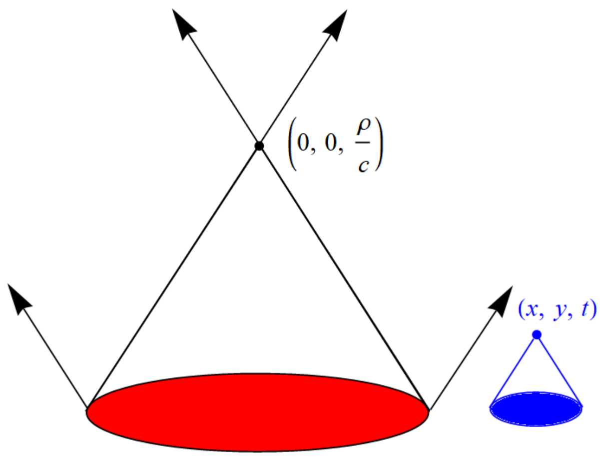


The purple area illustrated above represents the intersection of the blue disk centered at (x, y) with radius ct with the red disk. Letting $x_0 = r_0 \cos\theta_0$ and $y_0 = r_0 \sin\theta_0$, the double integral over this area is

$$\begin{aligned} u(x,y,t) &= \frac{1}{2\pi c}\iint\limits_{Q\cap T} \frac{A}{\sqrt{c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2}}\,dx_0\,dy_0 \\ &= \frac{1}{2\pi c}\int_0^{2\pi}\int_0^{\rho} \frac{A}{\sqrt{c^2t^2 - (r_0\cos\theta_0 - x)^2 - (r_0\sin\theta_0 - y)^2}}\,(r_0\,dr_0\,d\theta_0) \\ &= \frac{A}{2\pi c}\int_0^{2\pi}\int_0^{\rho} \frac{r_0}{\sqrt{c^2t^2 - (r_0\cos\theta_0 - x)^2 - (r_0\sin\theta_0 - y)^2}}\,dr_0\,d\theta_0. \end{aligned}$$

This formula for u is valid at points in space-time where $ct > \rho + r$, or $r < ct - \rho$. u will now be calculated in the region outside the cone right above the x_0y_0-plane.

The Third Region

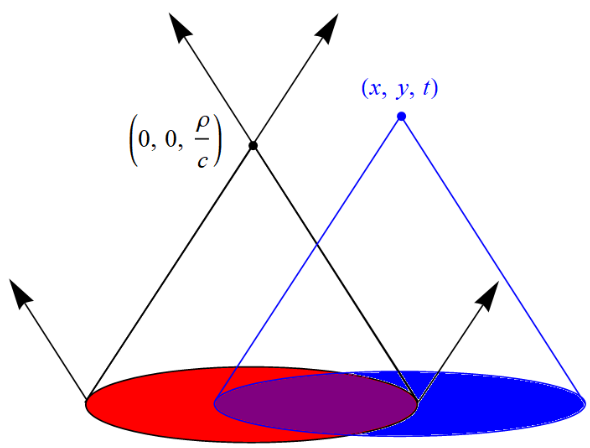


The blue disk illustrated above is centered at (x, y) with radius ct. Since it and the red disk are completely separate, the intersection is the empty set. Consequently,

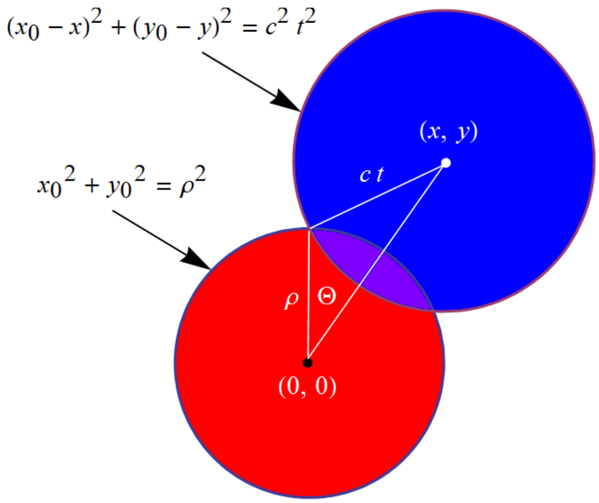
$$\begin{aligned} u(x, y, t) &= \frac{1}{2\pi c} \iint\limits_{Q \cap T} \frac{A}{\sqrt{c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2}}\, dx_0\, dy_0 \\ &= 0. \end{aligned}$$

This formula for u is valid at points in space-time where $r > \rho + ct$. u will now be calculated in the last region outside the cone.

The Fourth Region



The purple area illustrated above represents the intersection of the blue disk centered at (x, y) with radius ct with the red disk. Our aim now is to find the equations of the bounding curves and the angles at which the blue and red circles intersect.



$$
\begin{aligned}
&x_0^2 + y_0^2 = \rho^2 && (x_0 - x)^2 + (y_0 - y)^2 = c^2t^2 \\
&r_0^2 = \rho^2 && (r_0 \cos\theta_0 - x)^2 + (r_0 \sin\theta_0 - y)^2 = c^2t^2 \\
&r_0 = \rho && r_0^2 \cos^2\theta_0 - 2xr_0 \cos\theta_0 + x^2 + r_0^2 \sin^2\theta_0 - 2yr_0 \sin\theta_0 + y^2 = c^2t^2 \\
& && r_0^2 - 2(x\cos\theta_0 + y\sin\theta_0)r_0 = c^2t^2 - x^2 - y^2 \\
& && r_0^2 - 2(x\cos\theta_0 + y\sin\theta_0)r_0 + (x\cos\theta_0 + y\sin\theta_0)^2 = c^2t^2 - x^2 - y^2 + (x\cos\theta_0 + y\sin\theta_0)^2 \\
& && [r_0 - (x\cos\theta_0 + y\sin\theta_0)]^2 = c^2t^2 - x^2 - y^2 + (x\cos\theta_0 + y\sin\theta_0)^2 \\
& && r_0 - (x\cos\theta_0 + y\sin\theta_0) = \pm\sqrt{c^2t^2 - x^2 - y^2 + (x\cos\theta_0 + y\sin\theta_0)^2} \\
& && r_0 = x\cos\theta_0 + y\sin\theta_0 \pm \sqrt{c^2t^2 - x^2 - y^2 + (x\cos\theta_0 + y\sin\theta_0)^2} \\
& && r_0 = x\cos\theta_0 + y\sin\theta_0 - \sqrt{c^2t^2 - x^2 - y^2 + (x\cos\theta_0 + y\sin\theta_0)^2}
\end{aligned}
$$

The minus sign is chosen because we want the distance to the arc that is closest to the origin. To make the forthcoming formulas more compact, let $f(\theta_0)$ be defined as follows.

$$
\boxed{f(\theta_0) = x\cos\theta_0 + y\sin\theta_0 - \sqrt{c^2t^2 - x^2 - y^2 + (x\cos\theta_0 + y\sin\theta_0)^2}}
$$

Now use the law of cosines to relate Θ with the sides of the triangle above.

$$
c^2t^2 = \rho^2 + (x^2 + y^2) - 2\rho\sqrt{x^2 + y^2}\cos\Theta \quad \rightarrow \quad \Theta = \cos^{-1}\left(\frac{\rho^2 + x^2 + y^2 - c^2t^2}{2\rho\sqrt{x^2 + y^2}}\right)
$$

The angles at which the circles intersect are then

$$
\begin{aligned}
\tan^{-1}\left(\frac{y}{x}\right) - \Theta &= \tan^{-1}\left(\frac{y}{x}\right) - \cos^{-1}\left(\frac{\rho^2 + x^2 + y^2 - c^2t^2}{2\rho\sqrt{x^2 + y^2}}\right) \\
\tan^{-1}\left(\frac{y}{x}\right) + \Theta &= \tan^{-1}\left(\frac{y}{x}\right) + \cos^{-1}\left(\frac{\rho^2 + x^2 + y^2 - c^2t^2}{2\rho\sqrt{x^2 + y^2}}\right).
\end{aligned}
$$

Letting $x_0 = r_0\cos\theta_0$ and $y_0 = r_0\sin\theta_0$, the double integral over the purple area is

$$\begin{aligned}
u(x,y,t) &= \frac{1}{2\pi c}\iint\limits_{Q\cap T} \frac{A}{\sqrt{c^2t^2-(x_0-x)^2-(y_0-y)^2}}\,dx_0\,dy_0 \\
&= \frac{1}{2\pi c}\int_{\tan^{-1}\left(\frac{y}{x}\right)-\cos^{-1}\left(\frac{\rho^2+x^2+y^2-c^2t^2}{2\rho\sqrt{x^2+y^2}}\right)}^{\tan^{-1}\left(\frac{y}{x}\right)+\cos^{-1}\left(\frac{\rho^2+x^2+y^2-c^2t^2}{2\rho\sqrt{x^2+y^2}}\right)} \int_{f(\theta_0)}^{\rho} \frac{A}{\sqrt{c^2t^2-(r_0\cos\theta_0-x)^2-(r_0\sin\theta_0-y)^2}}\,(r_0\,dr_0\,d\theta_0) \\
&= \frac{A}{2\pi c}\int_{\tan^{-1}\left(\frac{y}{x}\right)-\cos^{-1}\left(\frac{\rho^2+x^2+y^2-c^2t^2}{2\rho\sqrt{x^2+y^2}}\right)}^{\tan^{-1}\left(\frac{y}{x}\right)+\cos^{-1}\left(\frac{\rho^2+x^2+y^2-c^2t^2}{2\rho\sqrt{x^2+y^2}}\right)} \int_{f(\theta_0)}^{\rho} \frac{r_0}{\sqrt{c^2t^2-(r_0\cos\theta_0-x)^2-(r_0\sin\theta_0-y)^2}}\,dr_0\,d\theta_0.
\end{aligned}$$

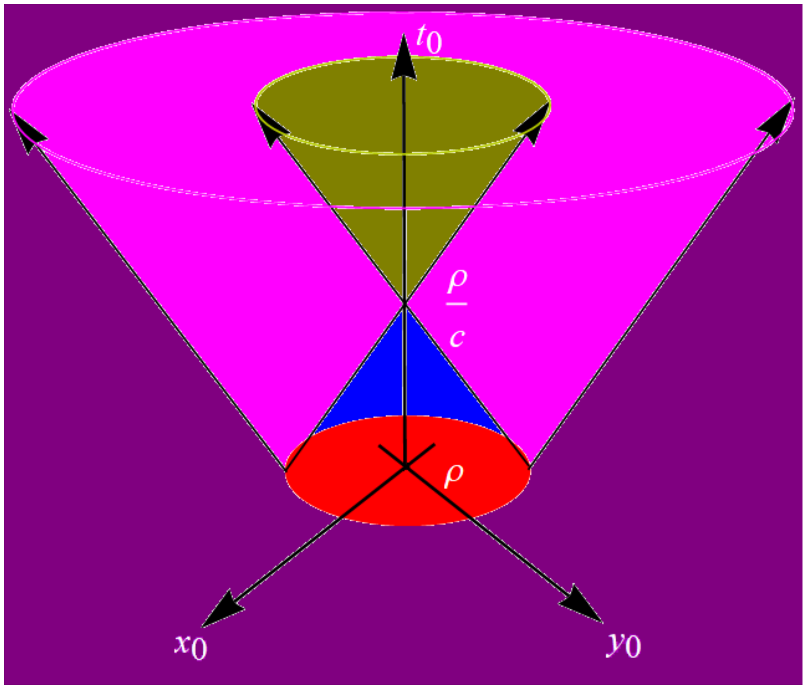
This formula for u is valid at points in space-time where $|\rho-ct|<r<\rho+ct$. To summarize the results, we have

$$u(x,y,t)=\begin{cases}
At & \text{if } r<\rho-ct \\
\dfrac{A}{2\pi c}\displaystyle\int_0^{2\pi}\int_0^{\rho}\frac{r_0}{\sqrt{c^2t^2-(r_0\cos\theta_0-x)^2-(r_0\sin\theta_0-y)^2}}\,dr_0\,d\theta_0 & \text{if } r<ct-\rho \\
0 & \text{if } r>\rho+ct \\
\dfrac{A}{2\pi c}\displaystyle\int_{\tan^{-1}\left(\frac{y}{x}\right)-\cos^{-1}\left(\frac{\rho^2+x^2+y^2-c^2t^2}{2\rho\sqrt{x^2+y^2}}\right)}^{\tan^{-1}\left(\frac{y}{x}\right)+\cos^{-1}\left(\frac{\rho^2+x^2+y^2-c^2t^2}{2\rho\sqrt{x^2+y^2}}\right)}\int_{f(\theta_0)}^{\rho}\frac{r_0}{\sqrt{c^2t^2-(r_0\cos\theta_0-x)^2-(r_0\sin\theta_0-y)^2}}\,dr_0\,d\theta_0 & \text{if } |\rho-ct|<r<\rho+ct
\end{cases}.$$

Therefore, replacing r with $\sqrt{x^2+y^2}$,

$$u(x,y,t)=\begin{cases}
At & \text{if } \sqrt{x^2+y^2}<\rho-ct \\
\dfrac{A}{2\pi c}\displaystyle\int_0^{2\pi}\int_0^{\rho}\frac{r_0}{\sqrt{c^2t^2-(r_0\cos\theta_0-x)^2-(r_0\sin\theta_0-y)^2}}\,dr_0\,d\theta_0 & \text{if } \sqrt{x^2+y^2}<ct-\rho \\
0 & \text{if } \sqrt{x^2+y^2}>\rho+ct \\
\dfrac{A}{2\pi c}\displaystyle\int_{\tan^{-1}\left(\frac{y}{x}\right)-\cos^{-1}\left(\frac{\rho^2+x^2+y^2-c^2t^2}{2\rho\sqrt{x^2+y^2}}\right)}^{\tan^{-1}\left(\frac{y}{x}\right)+\cos^{-1}\left(\frac{\rho^2+x^2+y^2-c^2t^2}{2\rho\sqrt{x^2+y^2}}\right)}\int_{f(\theta_0)}^{\rho}\frac{r_0}{\sqrt{c^2t^2-(r_0\cos\theta_0-x)^2-(r_0\sin\theta_0-y)^2}}\,dr_0\,d\theta_0 & \text{if } |\rho-ct|<\sqrt{x^2+y^2}<\rho+ct
\end{cases}.$$

Space-time is illustrated below; the solution to the wave equation in each region is labeled by color.



Part (b)

The solution at the origin is obtained by setting $x = 0$ and $y = 0$. Only the blue and olive solutions are applicable here.

$$u(0,0,t) = \begin{cases} At & \text{if } 0 < \rho - ct \\ \dfrac{A}{2\pi c}\displaystyle\int_0^{2\pi}\int_0^{\rho} \frac{r_0}{\sqrt{c^2t^2 - r_0^2\cos^2\theta_0 - r_0^2\sin^2\theta_0}}\,dr_0\,d\theta_0 & \text{if } 0 < ct - \rho \end{cases}$$

Use the same substitution as before, $v = c^2t^2 - r_0^2$ and $dv = -2r_0\,dr_0$, to evaluate the double integral.

$$\begin{aligned} \frac{A}{2\pi c}\int_0^{2\pi}\int_0^{\rho} \frac{r_0}{\sqrt{c^2t^2 - r_0^2\cos^2\theta_0 - r_0^2\sin^2\theta_0}}\,dr_0\,d\theta_0 &= \frac{A}{2\pi c}\int_0^{2\pi}\int_0^{\rho}\frac{r_0}{\sqrt{c^2t^2 - r_0^2}}\,dr_0\,d\theta_0 \\ &= \frac{A}{2\pi c}\left(\int_0^{2\pi} d\theta_0\right)\left[\int_{c^2t^2}^{c^2t^2-\rho^2}\frac{1}{\sqrt{v}}\left(-\frac{dv}{2}\right)\right] \\ &= \frac{A}{4\pi c}(2\pi)\int_{c^2t^2-\rho^2}^{c^2t^2} v^{-1/2}\,dv \\ &= \frac{A}{2c}(2v^{1/2})\Big|_{c^2t^2-\rho^2}^{c^2t^2} \end{aligned}$$

The double integral evaluates to

$$\begin{aligned}\frac{A}{2\pi c}\int_0^{2\pi}\int_0^{\rho}\frac{r_0}{\sqrt{c^2t^2 - r_0^2\cos^2\theta_0 - r_0^2\sin^2\theta_0}}\,dr_0\,d\theta_0 &= \frac{A}{c}(\sqrt{c^2t^2} - \sqrt{c^2t^2-\rho^2}) \\ &= \frac{A}{c}\left(ct - c\sqrt{t^2 - \frac{\rho^2}{c^2}}\right) \\ &= A\left(t - \sqrt{t^2 - \frac{\rho^2}{c^2}}\right),\end{aligned}$$

which means that

$$u(0,0,t) = \begin{cases} At & \text{if } t < \frac{\rho}{c} \\ A\left(t - \sqrt{t^2 - \frac{\rho^2}{c^2}}\right) & \text{if } t > \frac{\rho}{c}\end{cases}.$$

Note that substituting $t = \rho/c$ into the olive formula gives At, so the solution is continuous. The $<$ and $>$ symbols can be replaced with $\leq$ and $\geq$, respectively. Therefore,

$$u(0,0,t) = \begin{cases} At & \text{if } t \leq \frac{\rho}{c} \\ A\left(t - \sqrt{t^2 - \frac{\rho^2}{c^2}}\right) & \text{if } t \geq \frac{\rho}{c}\end{cases}.$$