Exercise 5

Verify the correctness of (13) for the example $u(x,y,z,t) = t^2$ and $f(x,y,z,t) \equiv 2$.

Solution

The initial value problem to solve is

$$\begin{aligned} &u_{tt} - c^2\nabla^2 u = 2, \quad -\infty < x, y, z < \infty,\ t > 0 \\ &u(x,y,z,0) = 0 \\ &u_t(x,y,z,0) = 0. \end{aligned}$$

Equation (13) in the text gives the solution for it.

$$u(\mathbf{x},t) = \frac{1}{4\pi c^2} \iiint\limits_{\{|\boldsymbol{\xi}-\mathbf{x}|\leq ct\}} \frac{f(\boldsymbol{\xi}, t - |\boldsymbol{\xi}-\mathbf{x}|/c)}{|\boldsymbol{\xi}-\mathbf{x}|}\, d\boldsymbol{\xi} \tag{13}$$

We will verify that this volume integral over the solid ball centered at (x,y,z) with radius ct yields $u = t^2$ when carried out. Set $f = 2$ and note that $\boldsymbol{\xi} - \mathbf{x} = \langle x_0 - x, y_0 - y, z_0 - z\rangle$ and $d\boldsymbol{\xi} = dx_0\, dy_0\, dz_0$.

$$= \frac{1}{4\pi c^2} \iiint\limits_{|\boldsymbol{\xi}-\mathbf{x}|\leq ct} \frac{2}{\sqrt{(x_0-x)^2 + (y_0-y)^2 + (z_0-z)^2}}\, dx_0\, dy_0\, dz_0$$

Write out the volume integral explicitly by using spherical coordinates $(\rho_0, \theta_0, \phi_0)$, where ϕ_0 represents the angle from the polar axis.

$$\begin{aligned} x_0 - x &= \rho_0 \sin\theta_0 \cos\phi_0 \\ y_0 - y &= \rho_0 \sin\theta_0 \sin\phi_0 \\ z_0 - z &= \rho_0 \cos\theta_0 \end{aligned}$$

The formula becomes

$$\begin{aligned} u(\mathbf{x},t) &= \frac{1}{4\pi c^2} \int_0^\pi \int_0^{2\pi} \int_0^{ct} \frac{2}{\sqrt{(\rho_0 \sin\theta_0\cos\phi_0)^2 + (\rho_0\sin\theta_0\sin\phi_0)^2 + (\rho_0\cos\theta_0)^2}}\, \rho_0^2 \sin\phi_0\, d\rho_0\, d\theta_0\, d\phi_0 \\ &= \frac{1}{4\pi c^2} \int_0^\pi \int_0^{2\pi} \int_0^{ct} 2\rho_0 \sin\phi_0\, d\rho_0\, d\theta_0\, d\phi_0 \\ &= \frac{1}{4\pi c^2} \left(\int_0^{ct} 2\rho_0\, d\rho_0\right) \left(\int_0^{2\pi} d\theta_0\right) \left(\int_0^\pi \sin\phi_0\, d\phi_0\right) \\ &= \frac{1}{4\pi c^2}(c^2t^2)(2\pi)(2) \\ &= t^2. \end{aligned}$$

Therefore, equation (13) is verified for this particular example.