Exercise 6

Show that the unique solution of (9) is expressible in terms of the source operator by the simple formula (11).

Solution

The source operator $\mathscr{S} = \mathscr{S}(t)$ that Mr. Strauss is referring to is defined by

$$\begin{aligned}\mathscr{S} f(x,y,z,t) &= \left[\frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} (\cdot)\, dS_0\right] f(x,y,z,t) \\ &= \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} f(x_0,y_0,z_0,t)\, dS_0.\end{aligned}$$

Comparing it to the formula of Kirchhoff and Poisson, it satisfies the homogeneous three-dimensional wave equation with one nonzero initial condition,

$$\begin{aligned}&\mathscr{S}_{tt} - c^2 \Delta \mathscr{S} = 0, \quad -\infty < x, y, z < \infty,\ t > 0 \\ &\mathscr{S}(0) = 0 \\ &\mathscr{S}_t(0) = I,\end{aligned}$$

where I denotes the identity operator. We will show that equation (11) in the text,

$$u(\mathbf{x},t) = \int_0^t \mathscr{S}(t-s) f(\mathbf{x},s)\, ds, \tag{11}$$

is the solution to the initial value problem in equation (9).

$$\begin{gathered} u_{tt} - c^2 \Delta u = f(\mathbf{x},t) \\ u(\mathbf{x},0) \equiv 0, \quad u_t(\mathbf{x},0) \equiv 0 \end{gathered} \tag{9}$$

Differentiate u with respect to t using the Leibnitz rule.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} \int_0^t \mathscr{S}(t-s) f(\mathbf{x},s)\, ds \\ &= \int_0^t \frac{\partial}{\partial t} \mathscr{S}(t-s) f(\mathbf{x},s)\, ds + \mathscr{S}(0) f(\mathbf{x},t) \cdot 1 - \mathscr{S}(t) f(\mathbf{x},0) \cdot 0 \\ &= \int_0^t \mathscr{S}_t(t-s) f(\mathbf{x},s)\, ds + \mathscr{S}(0) f(\mathbf{x},t) \\ &= \int_0^t \mathscr{S}_t(t-s) f(\mathbf{x},s)\, ds + (0) f(\mathbf{x},t) \\ &= \int_0^t \mathscr{S}_t(t-s) f(\mathbf{x},s)\, ds\end{aligned}$$

Differentiate u with respect to t once more.

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\int_0^t \mathscr{S}_t(t-s)f(\mathbf{x},s)\,ds \\
&= \int_0^t \frac{\partial}{\partial t}\mathscr{S}_t(t-s)f(\mathbf{x},s)\,ds + \mathscr{S}_t(0)f(\mathbf{x},t)\cdot 1 - \mathscr{S}(t)f(\mathbf{x},0)\cdot 0 \\
&= \int_0^t \mathscr{S}_{tt}(t-s)f(\mathbf{x},s)\,ds + \mathscr{S}_t(0)f(\mathbf{x},t) \\
&= \int_0^t \mathscr{S}_{tt}(t-s)f(\mathbf{x},s)\,ds + (I)f(\mathbf{x},t) \\
&= \int_0^t \mathscr{S}_{tt}(t-s)f(\mathbf{x},s)\,ds + f(\mathbf{x},t) \\
&= \int_0^t c^2\Delta\mathscr{S}(t-s)f(\mathbf{x},s)\,ds + f(\mathbf{x},t) \\
&= c^2\Delta\int_0^t \mathscr{S}(t-s)f(\mathbf{x},s)\,ds + f(\mathbf{x},t) \\
&= c^2\Delta u + f(\mathbf{x},t)
\end{aligned}$$

Subtracting both sides by $c^2\Delta u$, we conclude that equation (11) satisfies the inhomogeneous three-dimensional wave equation.

$$\frac{\partial^2 u}{\partial t^2} - c^2\Delta u = f(\mathbf{x},t)$$

Substitute $t = 0$ in u and $\partial u/\partial t$ to check the initial conditions.

$$u(\mathbf{x},0) = \int_0^0 \mathscr{S}(-s)f(\mathbf{x},s)\,ds = 0 \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \int_0^0 \mathscr{S}_t(-s)f(\mathbf{x},s)\,ds = 0$$

Therefore, equation (11) solves the initial value problem in equation (9).

Uniqueness

Here we will show that the solution to

$$\begin{gathered} u_{tt} - c^2 \Delta u = f(\mathbf{x}, t) \\ u(\mathbf{x}, 0) \equiv 0, \quad u_t(\mathbf{x}, 0) \equiv 0 \end{gathered}$$

is unique. Suppose that there is a second solution to the initial value problem.

$$\begin{gathered} v_{tt} - c^2 \Delta v = f(\mathbf{x}, t) \\ v(\mathbf{x}, 0) \equiv 0, \quad v_t(\mathbf{x}, 0) \equiv 0 \end{gathered}$$

Subtract both sides of each equation for v from those for u, respectively.

$$\begin{gathered} u_{tt} - c^2 \Delta u - (v_{tt} - c^2 \Delta v) = f(\mathbf{x}, t) - f(\mathbf{x}, t) \\ u(\mathbf{x}, 0) - v(\mathbf{x}, 0) \equiv 0, \quad u_t(\mathbf{x}, 0) - v_t(\mathbf{x}, 0) \equiv 0 \end{gathered}$$

$$\begin{gathered} u_{tt} - v_{tt} - c^2(\Delta u - \Delta v) = 0 \\ u(\mathbf{x}, 0) - v(\mathbf{x}, 0) \equiv 0, \quad u_t(\mathbf{x}, 0) - v_t(\mathbf{x}, 0) \equiv 0 \end{gathered}$$

$$\begin{gathered} (u - v)_{tt} - c^2 \Delta (u - v) = 0 \\ u(\mathbf{x}, 0) - v(\mathbf{x}, 0) \equiv 0, \quad u_t(\mathbf{x}, 0) - v_t(\mathbf{x}, 0) \equiv 0 \end{gathered}$$

Let $w(\mathbf{x}, t) = u(\mathbf{x}, t) - v(\mathbf{x}, t)$.

$$\begin{gathered} w_{tt} - c^2 \Delta w = 0 \\ w(\mathbf{x}, 0) \equiv 0, \quad w_t(\mathbf{x}, 0) \equiv 0 \end{gathered}$$

According to the formula of Kirchhoff and Poisson,

$$\begin{aligned} w(\mathbf{x}, t) &= \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} (0)\, dS_0\right] + \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} (0)\, dS_0 \\ &= 0, \end{aligned}$$

which means that the solutions are one and the same function.

$$u(\mathbf{x}, t) = v(\mathbf{x}, t)$$

Therefore, the solution to the initial value problem in equation (9) is unique.