Exercise 9

Simplify formula (13) for the solution of $u_{tt} - c^2 \Delta u = f(\mathbf{x}, t)$ in the special case that f is spherically symmetric [$f = f(r, t)$].

Solution

Begin by solving the three-dimensional wave equation with a spherically symmetric source.

$$u_{tt} - c^2 \Delta u = f(r, t), \quad -\infty < x, y, z < \infty, \ t > 0$$

Expand the Laplacian operator in spherical coordinates (r, θ, ϕ), where ϕ represents the angle from the polar axis.

$$u_{tt} - c^2 \left[u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \left(u_{\theta\theta} + (\cot\theta) u_\theta + \frac{1}{\sin^2\theta} u_{\phi\phi} \right) \right] = f(r, t)$$

Since f is spherically symmetric, u is spherically symmetric as well [$u = u(r, t)$], which means the angular derivatives of u vanish.

$$u_{tt} - c^2 \left(u_{rr} + \frac{2}{r} u_r \right) = f(r, t), \quad 0 < r < \infty, \ t > 0$$

Make the substitution $v(r, t) = r u(r, t)$. Find the derivatives of u in terms of this new variable.

$$v_{tt} = r u_{tt} \quad \rightarrow \quad u_{tt} = \frac{v_{tt}}{r}$$
$$v_r = u + r u_r$$
$$v_{rr} = u_r + u_r + r u_{rr} = 2u_r + r u_{rr} \quad \rightarrow \quad u_{rr} + \frac{2}{r} u_r = \frac{v_{rr}}{r}$$

Consequently, the PDE that v satisfies is

$$\frac{v_{tt}}{r} - c^2 \frac{v_{rr}}{r} = f(r, t), \quad 0 < r < \infty, \ t > 0.$$

Multiply both sides by r.

$$v_{tt} - c^2 v_{rr} = r f(r, t)$$

As determined in Exercise 12 of Section 3.4, the solution to this PDE is

$$v(r, t) = \begin{cases} \dfrac{1}{2c} \displaystyle\int_0^t \int_{|r - c(t - t_0)|}^{r + c(t - t_0)} r_0 f(r_0, t_0) \, dr_0 \, dt_0 & \text{if } r - ct < 0 \\ \dfrac{1}{2c} \displaystyle\int_0^t \int_{r - c(t - t_0)}^{r + c(t - t_0)} r_0 f(r_0, t_0) \, dr_0 \, dt_0 & \text{if } r - ct > 0 \end{cases}.$$

Therefore, dividing both sides by r,

$$u(r, t) = \begin{cases} \dfrac{1}{2cr} \displaystyle\int_0^t \int_{|r - c(t - t_0)|}^{r + c(t - t_0)} r_0 f(r_0, t_0) \, dr_0 \, dt_0 & \text{if } r - ct < 0 \\ \dfrac{1}{2cr} \displaystyle\int_0^t \int_{r - c(t - t_0)}^{r + c(t - t_0)} r_0 f(r_0, t_0) \, dr_0 \, dt_0 & \text{if } r - ct > 0 \end{cases}.$$

The aim in this exercise is to show that equation (13) reduces to this result in the case that f is spherically symmetric.

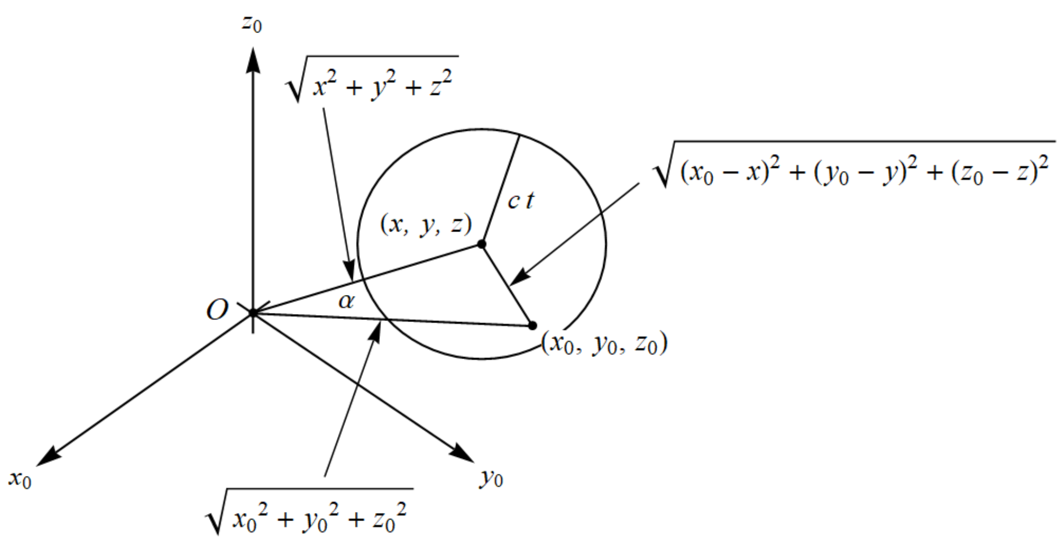
$$u(\mathbf{x}, t) = \frac{1}{4\pi c^2} \iiint_{\{|\boldsymbol{\xi}-\mathbf{x}| \le ct\}} \frac{f(\boldsymbol{\xi}, t - |\boldsymbol{\xi} - \mathbf{x}|/c)}{|\boldsymbol{\xi} - \mathbf{x}|} \, d\boldsymbol{\xi} \tag{13}$$

$$u(x, y, z, t) = \frac{1}{4\pi c^2} \iiint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2 \le c^2t^2}} \frac{f\left(x_0, y_0, z_0, t - \frac{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}{c}\right)}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}} \, dV_0$$

This volume integral is over the solid ball in $x_0y_0z_0$-space centered at (x, y, z) with radius ct.

Case I: $r - ct > 0$

Suppose first that $r^2 = x^2 + y^2 + z^2 > c^2t^2$.



Notice that the denominator of the integrand is the distance between (x_0, y_0, z_0) and (x, y, z). Use the law of cosines to relate the sides of the triangle with α.

$$(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 = (x_0^2 + y_0^2 + z_0^2) + (x^2 + y^2 + z^2) - 2\sqrt{x^2 + y^2 + z^2}\sqrt{x_0^2 + y_0^2 + z_0^2}\cos\alpha$$

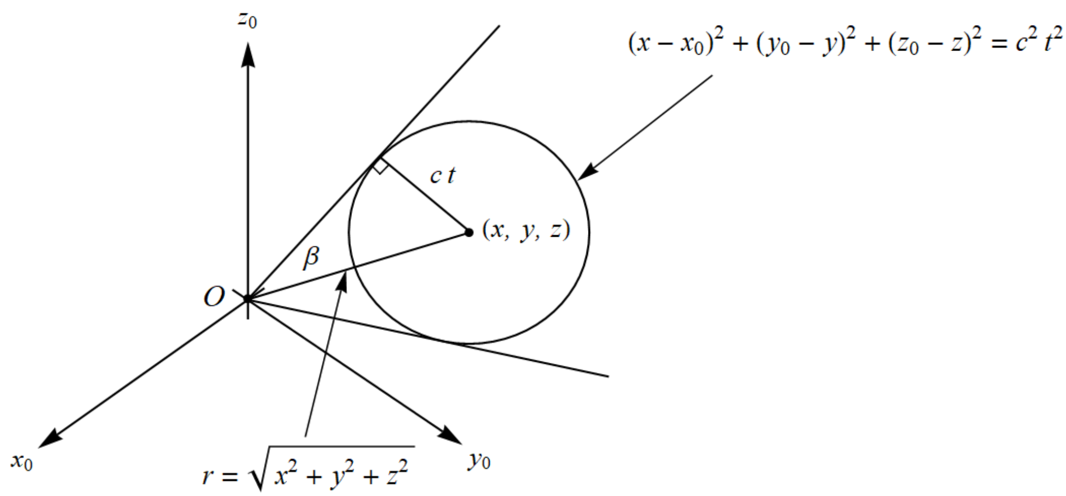
Take the square root of both sides.

$$\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2} = \sqrt{x_0^2 + y_0^2 + z_0^2 - 2\sqrt{x^2 + y^2 + z^2}\sqrt{x_0^2 + y_0^2 + z_0^2}\cos\alpha + x^2 + y^2 + z^2}$$

Substitute $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$ and $r = \sqrt{x^2 + y^2 + z^2}$.

$$\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2} = \sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}$$

Now determine the maximum polar angle β and the bounding curves of the ball.



According to the Pythagorean theorem, the length of the missing side of the triangle is $r\cos\beta = \sqrt{r^2 - c^2t^2}$. The bounding curves are

$$(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 = c^2t^2$$

$$r_0^2 - 2rr_0\cos\alpha + r^2 = c^2t^2$$

$$r_0^2 - (2r\cos\alpha)r_0 + (r^2 - c^2t^2) = 0$$

$$r_0 = \frac{2r\cos\alpha \pm \sqrt{4r^2\cos^2\alpha - 4(r^2 - c^2t^2)}}{2}$$

$$r_0 = r\cos\alpha \pm \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)}.$$

As a result of changing to spherical coordinates (r_0, θ_0, ϕ_0), choosing the polar axis to lie in the direction of (x, y, z), the solution for u becomes

$$u(r,t) = \frac{1}{4\pi c^2}\int_0^\beta \int_0^{2\pi} \int_{r\cos\alpha - \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)}}^{r\cos\alpha + \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)}} \frac{f\left(r_0, t - \frac{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}}{c}\right)}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}} r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0.$$

Orienting the polar axis as we have makes it so that $\alpha = \phi_0$.

$$u(r,t) = \frac{1}{4\pi c^2}\int_0^\beta \int_0^{2\pi} \int_{r\cos\phi_0 - \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}}^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}} \frac{f\left(r_0, t - \frac{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}{c}\right)}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0$$

Evaluate the integral in $d\theta_0$.

$$\begin{aligned} u(r,t) &= \frac{1}{4\pi c^2}\left(\int_0^{2\pi} d\theta_0\right) \int_0^\beta \int_{r\cos\phi_0 - \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}}^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}} \frac{f\left(r_0, t - \frac{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}{c}\right)}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} r_0^2 \sin\phi_0\, dr_0\, d\phi_0 \\ &= \frac{1}{2c^2}\int_0^\beta \int_{r\cos\phi_0 - \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}}^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}} \frac{f\left(r_0, t - \frac{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}{c}\right)}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} r_0^2 \sin\phi_0\, dr_0\, d\phi_0 \end{aligned}$$

Make the following substitution.

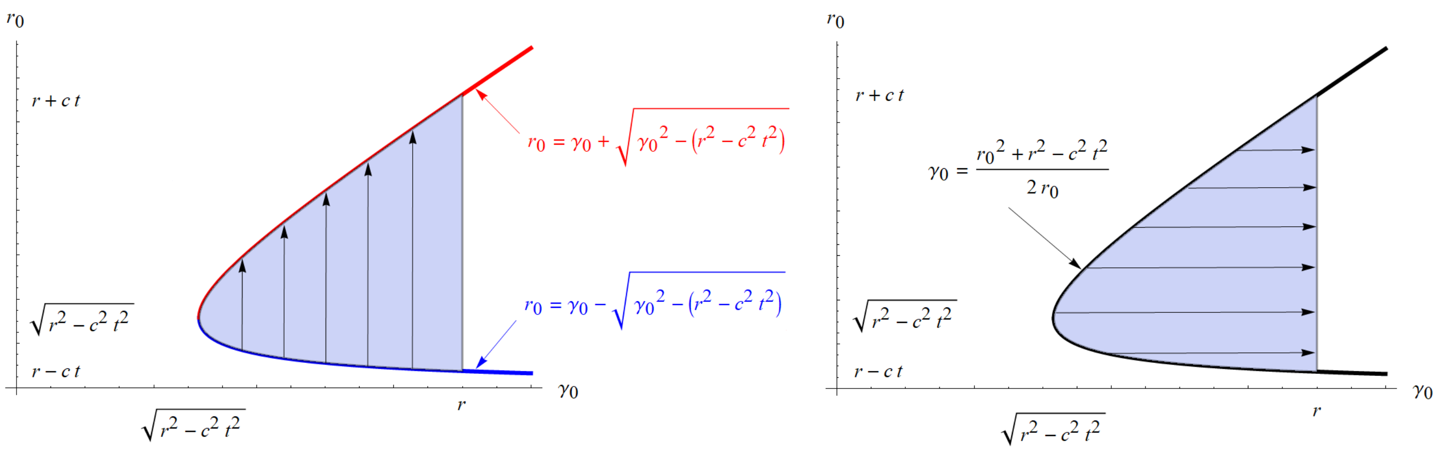
$$\gamma_0 = r\cos\phi_0$$

$$d\gamma_0 = -r\sin\phi_0\,d\phi_0 \quad \rightarrow \quad -\frac{d\gamma_0}{r} = \sin\phi_0\,d\phi_0$$

Consequently,

$$\begin{aligned} u(r,t) &= \frac{1}{2c^2}\int_r^{r\cos\beta}\int_{\gamma_0-\sqrt{\gamma_0^2-(r^2-c^2t^2)}}^{\gamma_0+\sqrt{\gamma_0^2-(r^2-c^2t^2)}} \frac{f\left(r_0, t-\frac{\sqrt{r_0^2-2r_0\gamma_0+r^2}}{c}\right)}{\sqrt{r_0^2-2r_0\gamma_0+r^2}} r_0^2\,dr_0\left(-\frac{d\gamma_0}{r}\right) \\ &= \frac{1}{2c^2 r}\int_{r\cos\beta}^{r}\int_{\gamma_0-\sqrt{\gamma_0^2-(r^2-c^2t^2)}}^{\gamma_0+\sqrt{\gamma_0^2-(r^2-c^2t^2)}} \frac{f\left(r_0, t-\frac{\sqrt{r_0^2-2r_0\gamma_0+r^2}}{c}\right)}{\sqrt{r_0^2-2r_0\gamma_0+r^2}} r_0^2\,dr_0\,d\gamma_0 \\ &= \frac{1}{2c^2 r}\int_{\sqrt{r^2-c^2t^2}}^{r}\int_{\gamma_0-\sqrt{\gamma_0^2-(r^2-c^2t^2)}}^{\gamma_0+\sqrt{\gamma_0^2-(r^2-c^2t^2)}} \frac{f\left(r_0, t-\frac{\sqrt{r_0^2-2r_0\gamma_0+r^2}}{c}\right)}{\sqrt{r_0^2-2r_0\gamma_0+r^2}} r_0^2\,dr_0\,d\gamma_0. \end{aligned}$$

The current mode of integration in the $\gamma_0 r_0$-plane is shown below on the left.



Integrate over the domain as shown on the right to switch the order of integration.

$$u(r,t) = \frac{1}{2c^2 r}\int_{r-ct}^{r+ct}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r} \frac{f\left(r_0, t-\frac{\sqrt{r_0^2-2r_0\gamma_0+r^2}}{c}\right)}{\sqrt{r_0^2-2r_0\gamma_0+r^2}} r_0^2\,d\gamma_0\,dr_0$$

Now make another substitution.

$$s_0 = r_0^2 - 2r_0\gamma_0 + r^2$$

$$ds_0 = -2r_0\,d\gamma_0 \quad \rightarrow \quad -\frac{ds_0}{2} = r_0\,d\gamma_0$$

Consequently,

$$\begin{aligned} u(r,t) &= \frac{1}{2c^2 r}\int_{r-ct}^{r+ct}\int_{c^2t^2}^{r_0^2-2rr_0+r^2} \frac{f\left(r_0, t-\frac{\sqrt{s_0}}{c}\right)}{\sqrt{s_0}} r_0\left(-\frac{ds_0}{2}\right) dr_0 \\ &= \frac{1}{2cr}\int_{r-ct}^{r+ct}\int_{(r_0-r)^2}^{c^2t^2} \frac{f\left(r_0, t-\frac{\sqrt{s_0}}{c}\right)}{2c\sqrt{s_0}} r_0\,ds_0\,dr_0. \end{aligned}$$

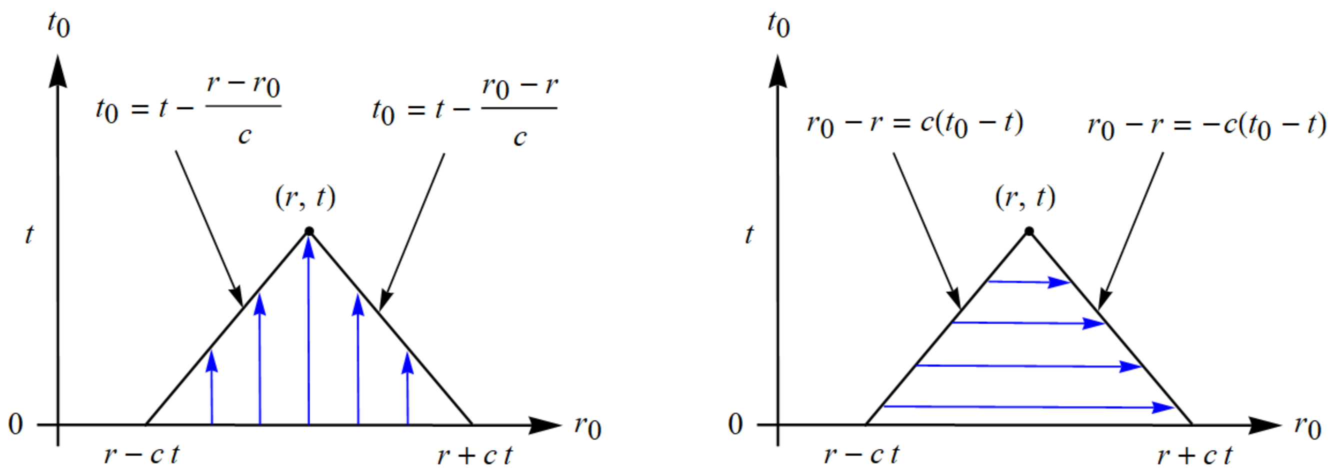
Make a final substitution.

$$t_0 = t - \frac{\sqrt{s_0}}{c}$$
$$dt_0 = -\frac{ds_0}{2c\sqrt{s_0}} \quad \to \quad -dt_0 = \frac{ds_0}{2c\sqrt{s_0}}$$

As a result,

$$\begin{aligned} u(r,t) &= \frac{1}{2cr}\int_{r-ct}^{r+ct}\int_{t-\frac{|r_0-r|}{c}}^{0} f(r_0,t_0)r_0\,(-dt_0)\,dr_0 \\ &= \frac{1}{2cr}\int_{r-ct}^{r+ct}\int_{0}^{t-\frac{|r_0-r|}{c}} f(r_0,t_0)r_0\,dt_0\,dr_0. \end{aligned}$$

The current mode of integration in the r_0t_0-plane is shown below on the left.



Integrate over the domain as shown on the right to switch the order of integration.

$$u(r,t) = \frac{1}{2cr}\int_0^t \int_{r+c(t_0-t)}^{r-c(t_0-t)} f(r_0,t_0)r_0\,dr_0\,dt_0$$

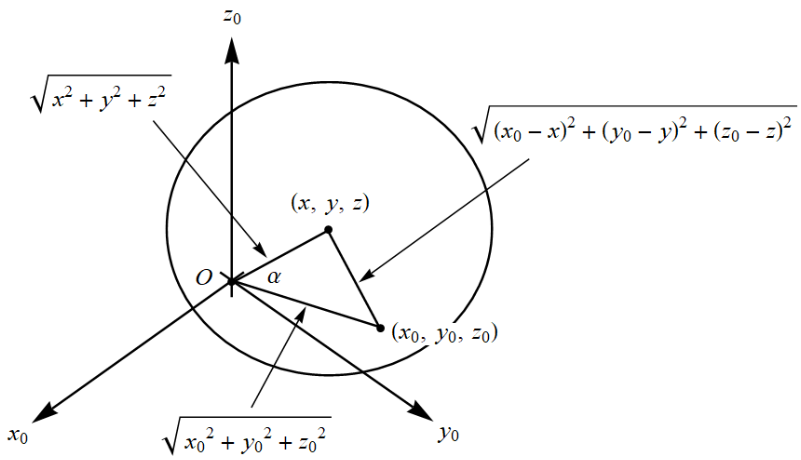
Therefore,

$$u(r,t) = \frac{1}{2cr}\int_0^t \int_{r-c(t-t_0)}^{r+c(t-t_0)} f(r_0,t_0)r_0\,dr_0\,dt_0,$$

for the case that f is spherically symmetric and $r - ct > 0$.

Case II: $r - ct < 0$

Suppose secondly that $r^2 = x^2 + y^2 + z^2 < c^2t^2$.



Notice that the denominator of the integrand is the distance between (x_0, y_0, z_0) and (x, y, z). Use the law of cosines to relate the sides of the triangle with α.

$$(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 = (x_0^2 + y_0^2 + z_0^2) + (x^2 + y^2 + z^2) - 2\sqrt{x^2 + y^2 + z^2}\sqrt{x_0^2 + y_0^2 + z_0^2}\cos\alpha$$

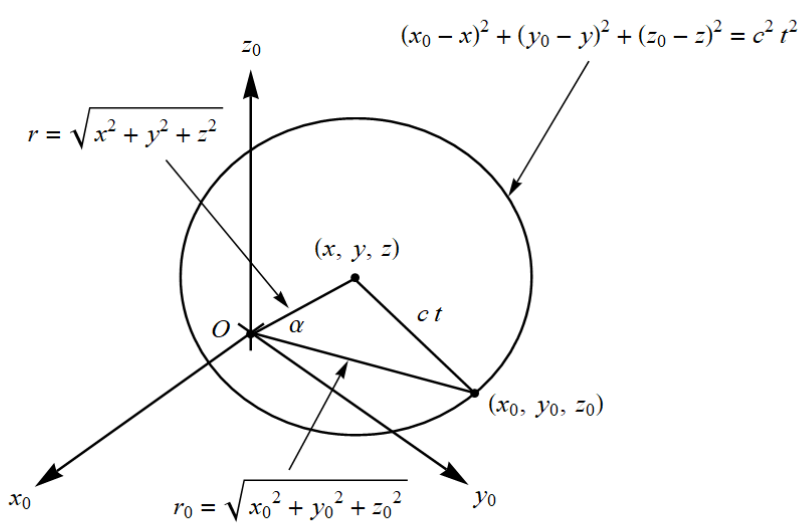
Take the square root of both sides.

$$\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2} = \sqrt{x_0^2 + y_0^2 + z_0^2 - 2\sqrt{x^2 + y^2 + z^2}\sqrt{x_0^2 + y_0^2 + z_0^2}\cos\alpha + x^2 + y^2 + z^2}$$

Substitute $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$ and $r = \sqrt{x^2 + y^2 + z^2}$.

$$\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2} = \sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}$$

Now determine the bounding curves of the ball.



The bounding curves are

$$\begin{gathered}
(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 = c^2t^2 \\
r_0^2 - 2rr_0 \cos\alpha + r^2 = c^2t^2 \\
r_0^2 - (2r\cos\alpha)r_0 - (c^2t^2 - r^2) = 0 \\
r_0 = \frac{2r\cos\alpha \pm \sqrt{4r^2\cos^2\alpha + 4(c^2t^2 - r^2)}}{2} \\
r_0 = r\cos\alpha \pm \sqrt{r^2\cos^2\alpha + (c^2t^2 - r^2)}.
\end{gathered}$$

We choose the positive sign because the distance from the origin to (x_0, y_0, z_0) must be positive.

$$r_0 = r\cos\alpha + \sqrt{r^2\cos^2\alpha + (c^2t^2 - r^2)}.$$

As a result of changing to spherical coordinates (r_0, θ_0, ϕ_0), the solution for u becomes

$$u(r,t) = \frac{1}{4\pi c^2}\int_0^\pi \int_0^{2\pi} \int_0^{r\cos\alpha + \sqrt{r^2\cos^2\alpha + (c^2t^2 - r^2)}} \frac{f\left(r_0, t - \frac{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}}{c}\right)}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}} r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0.$$

Orient the polar axis in the direction of (x, y, z) so that $\alpha = \phi_0$.

$$u(r,t) = \frac{1}{4\pi c^2}\int_0^\pi \int_0^{2\pi} \int_0^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{f\left(r_0, t - \frac{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}{c}\right)}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0$$

Evaluate the integral in $d\theta_0$.

$$\begin{aligned}
u(r,t) &= \frac{1}{4\pi c^2}\left(\int_0^{2\pi} d\theta_0\right)\int_0^\pi \int_0^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{f\left(r_0, t - \frac{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}{c}\right)}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0 \\
&= \frac{1}{2c^2}\int_0^\pi \int_0^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{f\left(r_0, t - \frac{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}{c}\right)}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0
\end{aligned}$$

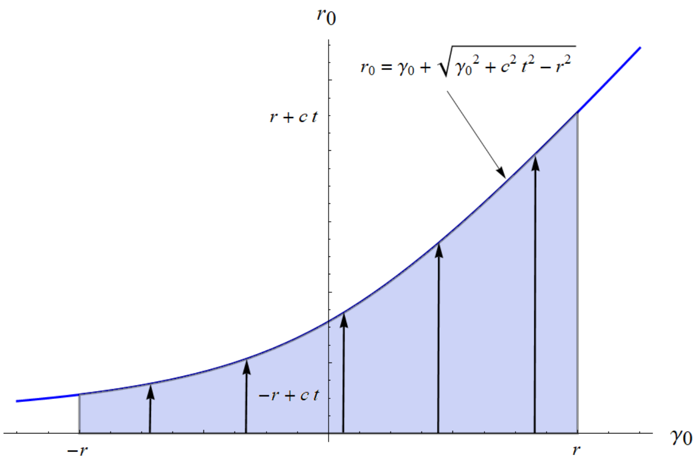
Make the following substitution.

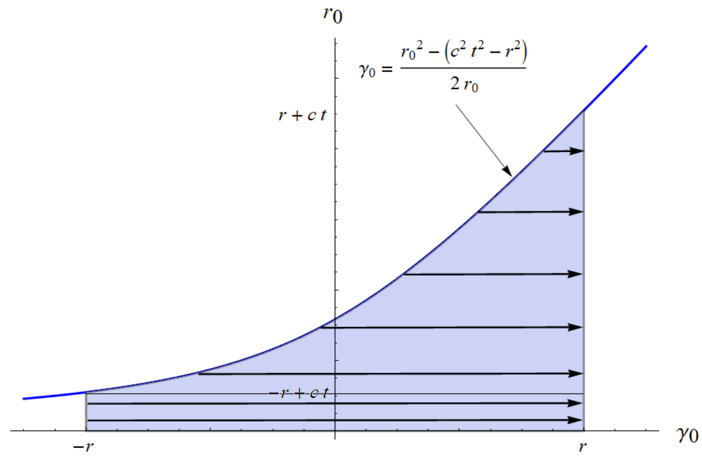
$$\begin{aligned}
\gamma_0 &= r\cos\phi_0 \\
d\gamma_0 &= -r\sin\phi_0 \, d\phi_0 \quad \rightarrow \quad -\frac{d\gamma_0}{r} = \sin\phi_0 \, d\phi_0
\end{aligned}$$

Consequently,

$$\begin{aligned}
u(r,t) &= \frac{1}{2c^2}\int_r^{-r} \int_0^{\gamma_0 + \sqrt{\gamma_0^2 + (c^2t^2 - r^2)}} \frac{f\left(r_0, t - \frac{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}{c}\right)}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} r_0^2 \, dr_0 \left(-\frac{d\gamma_0}{r}\right) \\
&= \frac{1}{2c^2 r}\int_{-r}^{r} \int_0^{\gamma_0 + \sqrt{\gamma_0^2 + (c^2t^2 - r^2)}} \frac{f\left(r_0, t - \frac{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}{c}\right)}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} r_0^2 \, dr_0 \, d\gamma_0.
\end{aligned}$$

The current mode of integration in the $\gamma_0 r_0$-plane is shown below on the left.





Integrate over the domain as shown on the right to switch the order of integration.

$$u(r,t) = \frac{1}{2c^2 r}\left[\int_0^{-r+ct}\int_{-r}^{r} \frac{f\left(r_0, t - \frac{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}{c}\right)}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} r_0^2 \, d\gamma_0 \, dr_0 \right.$$

$$\left. + \int_{-r+ct}^{r+ct}\int_{\frac{r_0^2 - (c^2t^2 - r^2)}{2r_0}}^{r} \frac{f\left(r_0, t - \frac{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}{c}\right)}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} r_0^2 \, d\gamma_0 \, dr_0 \right]$$

Now make another substitution in both integrals.

$$s_0 = r_0^2 - 2r_0\gamma_0 + r^2$$

$$ds_0 = -2r_0 \, d\gamma_0 \quad \rightarrow \quad -\frac{ds_0}{2} = r_0 \, d\gamma_0$$

Consequently,

$$u(r,t) = \frac{1}{2c^2 r}\left[\int_0^{-r+ct}\int_{r_0^2 + 2r_0 r + r^2}^{r_0^2 - 2r_0 r + r^2} \frac{f\left(r_0, t - \frac{\sqrt{s_0}}{c}\right)}{\sqrt{s_0}} r_0 \left(-\frac{ds_0}{2}\right) dr_0 \right.$$

$$\left. + \int_{-r+ct}^{r+ct}\int_{c^2t^2}^{r_0^2 - 2r_0 r + r^2} \frac{f\left(r_0, t - \frac{\sqrt{s_0}}{c}\right)}{\sqrt{s_0}} r_0 \left(-\frac{ds_0}{2}\right) dr_0 \right]$$

$$= \frac{1}{2cr}\left[\int_0^{-r+ct}\int_{(r_0 - r)^2}^{(r_0 + r)^2} \frac{f\left(r_0, t - \frac{\sqrt{s_0}}{c}\right)}{2c\sqrt{s_0}} r_0 \, ds_0 \, dr_0 \right.$$

$$\left. + \int_{-r+ct}^{r+ct}\int_{(r_0 - r)^2}^{c^2t^2} \frac{f\left(r_0, t - \frac{\sqrt{s_0}}{c}\right)}{2c\sqrt{s_0}} r_0 \, ds_0 \, dr_0 \right].$$

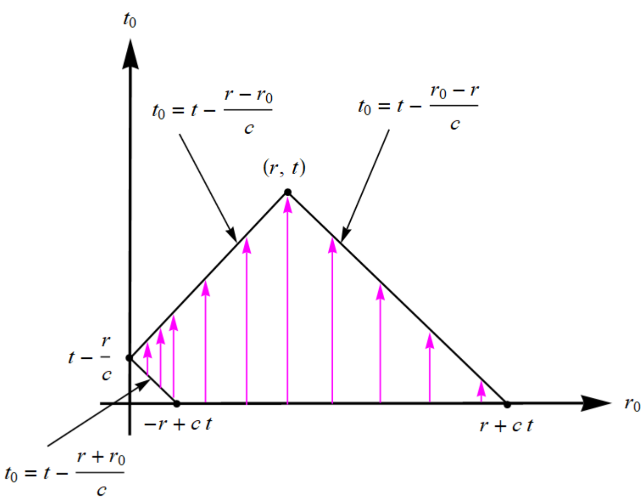
Make a final substitution in both integrals.

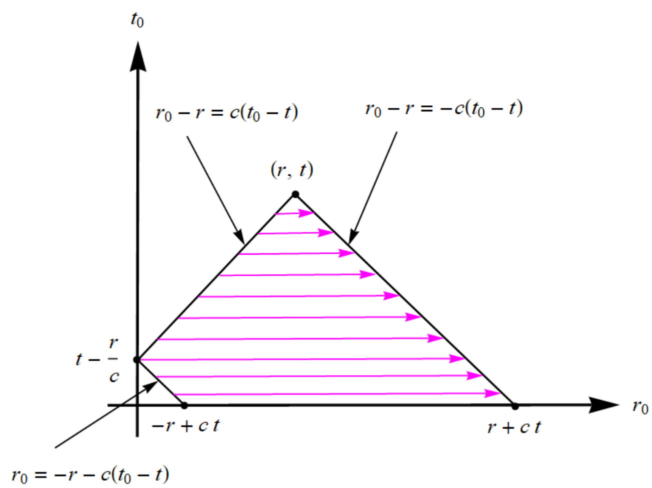
$$t_0 = t - \frac{\sqrt{s_0}}{c}$$
$$dt_0 = -\frac{ds_0}{2c\sqrt{s_0}} \quad \to \quad -dt_0 = \frac{ds_0}{2c\sqrt{s_0}}$$

As a result,

$$\begin{aligned} u(r,t) &= \frac{1}{2cr}\left[\int_0^{-r+ct}\int_{t-\frac{|r_0-r|}{c}}^{t-\frac{r_0+r}{c}} f(r_0,t_0)r_0(-dt_0)\,dr_0 + \int_{-r+ct}^{r+ct}\int_{t-\frac{|r_0-r|}{c}}^{0} f(r_0,t_0)r_0(-dt_0)\,dr_0\right] \\ &= \frac{1}{2cr}\left[\int_0^{-r+ct}\int_{t-\frac{r_0+r}{c}}^{t-\frac{|r_0-r|}{c}} f(r_0,t_0)r_0\,dt_0\,dr_0 + \int_{-r+ct}^{r+ct}\int_0^{t-\frac{|r_0-r|}{c}} f(r_0,t_0)r_0\,dt_0\,dr_0\right]. \end{aligned}$$

The current mode of integration in the r_0t_0-plane is shown below on the left.





Integrate over the domain as shown on the right to switch the order of integration.

$$\begin{aligned} u(r,t) &= \frac{1}{2cr}\left[\int_0^{t-\frac{r}{c}}\int_{-r-c(t_0-t)}^{r-c(t_0-t)} f(r_0,t_0)r_0\,dr_0\,dt_0 + \int_{t-\frac{r}{c}}^{t}\int_{r+c(t_0-t)}^{r-c(t_0-t)} f(r_0,t_0)\,r_0\,dr_0\,dt_0\right] \\ &= \frac{1}{2cr}\left[\int_0^{t-\frac{r}{c}}\int_{-[r-c(t-t_0)]}^{r+c(t-t_0)} f(r_0,t_0)r_0\,dr_0\,dt_0 + \int_{t-\frac{r}{c}}^{t}\int_{r-c(t-t_0)}^{r+c(t-t_0)} f(r_0,t_0)\,r_0\,dr_0\,dt_0\right] \end{aligned}$$

Therefore,

$$u(r,t) = \frac{1}{2cr}\int_0^t\int_{|r-c(t-t_0)|}^{r+c(t-t_0)} f(r_0,t_0)r_0\,dr_0\,dt_0,$$

for the case that f is spherically symmetric and $r - ct < 0$.