Exercise 1

Find a simple formula for the solution of the three-dimensional diffusion equation with $\phi(x, y, z) = xy^2z$. (*Hint:* See Exercise 2.4.9 or 2.4.10.)

Solution

The initial value problem to solve is

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u, \quad -\infty < x, y, z < \infty, \ t > 0$$
$$u(x, y, z, 0) = \phi(x, y, z).$$

Because the diffusion equation is linear and the three spatial variables go from $-\infty$ to ∞, the triple Fourier transform can be applied to solve it. Here we define the triple Fourier transform of a function $u(x, y, z, t)$ as

$$\mathcal{F}\{u(x, y, z, t)\} = U(k, l, m, t) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(kx+ly+mz)} u(x, y, z, t)\, dx\, dy\, dz.$$

As a result, the derivatives of u with respect to x and y and z and t transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n U(k, l, m, t)$$
$$\mathcal{F}\left\{\frac{\partial^n u}{\partial y^n}\right\} = (il)^n U(k, l, m, t)$$
$$\mathcal{F}\left\{\frac{\partial^n u}{\partial z^n}\right\} = (im)^n U(k, l, m, t)$$
$$\mathcal{F}\left\{\frac{\partial^n u}{\partial t^n}\right\} = \frac{d^n U}{dt^n}$$

Take the triple Fourier transform of both sides of the diffusion equation

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} &= \mathcal{F}\{\kappa \nabla^2 u\} \\
&= \mathcal{F}\left\{\kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\right\} \\
&= \kappa \left(\mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathcal{F}\left\{\frac{\partial^2 u}{\partial y^2}\right\} + \mathcal{F}\left\{\frac{\partial^2 u}{\partial z^2}\right\}\right)
\end{aligned}$$

and its initial condition.

$$\mathcal{F}\{u(x, y, z, 0)\} = \mathcal{F}\{\phi(x, y, z)\} \quad \to \quad U(k, l, m, 0) = \Phi(k, l, m)$$

Transform the partial derivatives with the formulas above.

$$\begin{aligned}
\frac{dU}{dt} &= \kappa[(ik)^2 U(k, l, m, t) + (il)^2 U(k, l, m, t) + (im)^2 U(k, l, m, t)] \\
&= \kappa(-k^2 U - l^2 U - m^2 U) \\
&= -\kappa(k^2 + l^2 + m^2) U
\end{aligned}$$

With the help of the Fourier transform, the three-dimensional diffusion equation that we started with has been reduced to a first-order ODE. The general solution for it can be written in terms of the exponential function.

$$U(k,l,m,t) = A(k,l,m)e^{-\kappa(k^2+l^2+m^2)t}$$

Apply the transformed initial condition to determine $A(k,l,m)$.

$$U(k,l,m,0) = A(k,l,m) = \Phi(k,l,m)$$

Consequently,

$$U(k,l,m,t) = \Phi(k,l,m)e^{-\kappa(k^2+l^2+m^2)t}.$$

Now that U has been determined, the goal is to obtain u by taking the inverse triple Fourier transform.

$$\begin{aligned} u(x,y,z,t) &= \mathcal{F}^{-1}\{U(k,l,m,t)\} \\ &= \mathcal{F}^{-1}\{\Phi(k,l,m)e^{-\kappa(k^2+l^2+m^2)t}\} \end{aligned}$$

Since we are taking the inverse triple Fourier transform of a product, we can apply the convolution theorem, which says that

$$\mathcal{F}^{-1}\{F(k,l,m)G(k,l,m)\} = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-k,y-l,z-m)g(k,l,m)\,dk\,dl\,dm.$$

What this means is that we can find the inverse transforms of the individual functions making up U and express the answer for u as a triple integral. We will now find the inverse triple Fourier transform of the exponential function using the definition.

$$\begin{aligned} \mathcal{F}^{-1}\{e^{-\kappa(k^2+l^2+m^2)t}\} &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(kx+ly+mz)}e^{-\kappa(k^2+l^2+m^2)t}\,dk\,dl\,dm \\ &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\kappa k^2t+ikx}e^{-\kappa l^2t+ily}e^{-\kappa m^2t+imz}\,dk\,dl\,dm \\ &= \frac{1}{(2\pi)^{3/2}}\left(\int_{-\infty}^{\infty} e^{-\kappa k^2t+ikx}\,dk\right)\left(\int_{-\infty}^{\infty} e^{-\kappa l^2t+ily}\,dl\right)\left(\int_{-\infty}^{\infty} e^{-\kappa m^2t+imz}\,dm\right) \end{aligned}$$

Complete the square in the first exponent.

$$\begin{aligned} -\kappa k^2t+ikx &= -\kappa t\left(k^2-\frac{ikx}{\kappa t}\right) \\ &= -\kappa t\left(k^2-\frac{ikx}{\kappa t}+\frac{i^2x^2}{4\kappa^2t^2}\right)+\frac{i^2x^2}{4\kappa t} \\ &= -\kappa t\left(k-\frac{ix}{2\kappa t}\right)^2-\frac{x^2}{4\kappa t} \end{aligned}$$

Similarly, the other two exponents become

$$\begin{aligned} -\kappa l^2t+ily &= -\kappa t\left(l-\frac{iy}{2\kappa t}\right)^2-\frac{y^2}{4\kappa t} \\ -\kappa m^2t+imz &= -\kappa t\left(m-\frac{iz}{2\kappa t}\right)^2-\frac{z^2}{4\kappa t}. \end{aligned}$$

As a result, the inverse transform of the exponential function becomes

$$\begin{aligned}\mathcal{F}^{-1}\{e^{-\kappa(k^2+l^2+m^2)t}\} &= \frac{1}{(2\pi)^{3/2}}\left\{\int_{-\infty}^{\infty}\exp\left[-\kappa t\left(k-\frac{ix}{2\kappa t}\right)^2-\frac{x^2}{4\kappa t}\right]dk\right\}\\ &\quad\times\left\{\int_{-\infty}^{\infty}\exp\left[-\kappa t\left(l-\frac{iy}{2\kappa t}\right)^2-\frac{y^2}{4\kappa t}\right]dl\right\}\\ &\quad\times\left\{\int_{-\infty}^{\infty}\exp\left[-\kappa t\left(m-\frac{iz}{2\kappa t}\right)^2-\frac{z^2}{4\kappa t}\right]dm\right\}\\ &= \frac{1}{(2\pi)^{3/2}}\left\{\exp\left(-\frac{x^2}{4\kappa t}\right)\int_{-\infty}^{\infty}\exp\left[-\kappa t\left(k-\frac{ix}{2\kappa t}\right)^2\right]dk\right\}\\ &\quad\times\left\{\exp\left(-\frac{y^2}{4\kappa t}\right)\int_{-\infty}^{\infty}\exp\left[-\kappa t\left(l-\frac{iy}{2\kappa t}\right)^2\right]dl\right\}\\ &\quad\times\left\{\exp\left(-\frac{z^2}{4\kappa t}\right)\int_{-\infty}^{\infty}\exp\left[-\kappa t\left(m-\frac{iz}{2\kappa t}\right)^2\right]dm\right\}.\end{aligned}$$

Make use of the integration formula,

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}},$$

to evaluate the integrals.

$$\begin{aligned}\mathcal{F}^{-1}\{e^{-\kappa(k^2+l^2+m^2)t}\} &= \frac{1}{(2\pi)^{3/2}}\left\{\exp\left(-\frac{x^2}{4\kappa t}\right)\left(\sqrt{\frac{\pi}{\kappa t}}\right)\right\}\\ &\quad\times\left\{\exp\left(-\frac{y^2}{4\kappa t}\right)\left(\sqrt{\frac{\pi}{\kappa t}}\right)\right\}\\ &\quad\times\left\{\exp\left(-\frac{z^2}{4\kappa t}\right)\left(\sqrt{\frac{\pi}{\kappa t}}\right)\right\}\\ &= \frac{1}{(2\kappa t)^{3/2}}\exp\left(-\frac{x^2+y^2+z^2}{4\kappa t}\right)\end{aligned}$$

The convolution theorem can finally be applied to determine $u(x, y, z, t)$.

$$u(x,y,z,t) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(2\kappa t)^{3/2}}\exp\left[-\frac{(x-k)^2+(y-l)^2+(z-m)^2}{4\kappa t}\right]\phi(k,l,m)\,dk\,dl\,dm$$

Therefore,

$$\boxed{u(x,y,z,t) = \frac{1}{(4\pi\kappa t)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{(x-k)^2+(y-l)^2+(z-m)^2}{4\kappa t}\right]\phi(k,l,m)\,dk\,dl\,dm.}$$

This solution can be written compactly as

$$u(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G_3(x-k,y-l,z-m,t)\phi(k,l,m)\,dk\,dl\,dm,$$

where

$$G_3(x,y,z,t) = \frac{1}{(4\pi\kappa t)^{3/2}} \exp\left(-\frac{x^2+y^2+z^2}{4\kappa t}\right)$$

is the Green's function for the three-dimensional diffusion equation. It just so happens to be the product of three one-dimensional Green's functions because the exponential function can be split up.

$$\begin{aligned} G_3(x,y,z,t) &= \left[\frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right)\right]\left[\frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{y^2}{4\kappa t}\right)\right]\left[\frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{z^2}{4\kappa t}\right)\right] \\ &= G_1(x,t)G_1(y,t)G_1(z,t) \end{aligned}$$

Note that the Green's function for the diffusion equation is an analytic function, whereas the Green's function for the wave equation is a generalized function. This is generally how it is for elliptic and hyperbolic PDEs, respectively. Getting on with the exercise, $\phi(x,y,z) = xy^2z$, so the boxed solution becomes

$$\begin{aligned} u(x,y,z,t) &= \frac{1}{(4\pi\kappa t)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\frac{(x-k)^2+(y-l)^2+(z-m)^2}{4\kappa t}\right] kl^2m\,dk\,dl\,dm \\ &= \frac{1}{(4\pi\kappa t)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\frac{(x-k)^2}{4\kappa t}\right]\exp\left[-\frac{(y-l)^2}{4\kappa t}\right]\exp\left[-\frac{(z-m)^2}{4\kappa t}\right] kl^2m\,dk\,dl\,dm \\ &= \frac{1}{(4\pi\kappa t)^{3/2}} \left\{\int_{-\infty}^{\infty} k\exp\left[-\frac{(k-x)^2}{4\kappa t}\right]dk\right\}\left\{\int_{-\infty}^{\infty} l^2\exp\left[-\frac{(l-y)^2}{4\kappa t}\right]dl\right\}\int_{-\infty}^{\infty} m\exp\left[-\frac{(m-z)^2}{4\kappa t}\right]dm. \end{aligned}$$

Make the following substitutions.

$$\begin{aligned} q &= \frac{k-x}{\sqrt{4\kappa t}} & r &= \frac{l-y}{\sqrt{4\kappa t}} & s &= \frac{m-z}{\sqrt{4\kappa t}} \\ dq &= \frac{dk}{\sqrt{4\kappa t}} & dr &= \frac{dl}{\sqrt{4\kappa t}} & ds &= \frac{dm}{\sqrt{4\kappa t}} \end{aligned}$$

Consequently,

$$\begin{aligned} u(x,y,z,t) &= \frac{1}{\pi^{3/2}}\left[\int_{-\infty}^{\infty}(x+q\sqrt{4\kappa t})e^{-q^2}\,dq\right]\left[\int_{-\infty}^{\infty}(y+r\sqrt{4\kappa t})^2e^{-r^2}\,dr\right]\left[\int_{-\infty}^{\infty}(z+s\sqrt{4\kappa t})e^{-s^2}\,ds\right] \\ &= \frac{1}{\pi^{3/2}}\left(x\int_{-\infty}^{\infty}e^{-q^2}\,dq + \sqrt{4\kappa t}\int_{-\infty}^{\infty}qe^{-q^2}\,dq\right) \\ &\qquad \times\left(y^2\int_{-\infty}^{\infty}e^{-r^2}\,dr + 2y\sqrt{4\kappa t}\int_{-\infty}^{\infty}re^{-r^2}\,dr + 4\kappa t\int_{-\infty}^{\infty}r^2e^{-r^2}\,dr\right) \\ &\qquad \times\left(z\int_{-\infty}^{\infty}e^{-s^2}\,ds + \sqrt{4\kappa t}\int_{-\infty}^{\infty}se^{-s^2}\,ds\right) \\ &= \frac{1}{\pi^{3/2}}\left(x\cdot\sqrt{\pi}+\sqrt{4\kappa t}\cdot 0\right)\left(y^2\cdot\sqrt{\pi}+2y\sqrt{4\kappa t}\cdot 0+4\kappa t\cdot\frac{\sqrt{\pi}}{2}\right)\left(z\cdot\sqrt{\pi}+\sqrt{4\kappa t}\cdot 0\right) \\ &= \frac{1}{\pi^{3/2}}(x\sqrt{\pi})(y^2\sqrt{\pi}+2\kappa t\sqrt{\pi})(z\sqrt{\pi}). \end{aligned}$$

Therefore,

$$\boxed{u(x,y,z,t) = x(y^2+2\kappa t)z.}$$

Note that in evaluating the integrals, the formula

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

was used.

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = 0$$

is true because the integral of an odd function over a symmetric interval is zero. The final integral is

$$\begin{aligned}
\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx &= \lim_{a\to 1}\int_{-\infty}^{\infty} x^2 e^{-ax^2}\,dx \\
&= \lim_{a\to 1}\int_{-\infty}^{\infty}\left(-\frac{\partial}{\partial a}e^{-ax^2}\right)dx \\
&= -\lim_{a\to 1}\frac{d}{da}\int_{-\infty}^{\infty} e^{-ax^2}\,dx \\
&= -\lim_{a\to 1}\frac{d}{da}\left(\sqrt{\frac{\pi}{a}}\right) \\
&= -\lim_{a\to 1}\left(-\frac{1}{2}\sqrt{\frac{\pi}{a^3}}\right) \\
&= \frac{\sqrt{\pi}}{2}.
\end{aligned}$$