Exercise 4

Derive the first four Hermite polynomials from scratch.

Solution

The Hermite polynomials $H_k(x)$ are obtained by solving Hermite's differential equation,

$$w'' - 2xw' + (\lambda - 1)w = 0,$$

for positive odd integer values of λ: $\lambda = 2k + 1$.

The First Hermite Polynomial

To get the first Hermite polynomial $H_0(x)$, set $\lambda = 1$ and solve the resulting ODE.

$$w'' - 2xw' = 0$$

Multiply both sides by the integrating factor

$$I = \exp\left[\int^x (-2s)\,ds\right] = e^{-x^2}$$

to obtain

$$e^{-x^2}w'' - 2xe^{-x^2}w' = 0.$$

The left side can be written as $d/dx(Iw')$ by the product rule.

$$\frac{d}{dx}(e^{-x^2}w') = 0$$

Integrate both sides with respect to x.

$$e^{-x^2}w' = C_1$$

Multiply both sides by e^{x^2}.

$$w' = C_1e^{x^2}$$

Integrate both sides with respect to x.

$$w(x) = C_1\int^x e^{s^2}\,ds + C_2$$

Since we require a polynomial solution, we set $C_1 = 0$.

$$w(x) = C_2$$

By convention, the coefficient of the highest power of x is set to 2^0, so $C_2 = 1$. Therefore,

$$H_0(x) = 1.$$

The Second Hermite Polynomial

To get the second Hermite polynomial $H_1(x)$, set $\lambda = 3$ and solve the resulting ODE.

$$w'' - 2xw' + 2w = 0$$

We seek a power series solution for w.

$$w(x) = \sum_{n=0}^{\infty} a_n x^n \quad \to \quad w'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \to \quad w''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these formulas into the ODE.

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=0}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

The first sum can be started from $n = 2$.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Substitute $j = n - 2$ in the first sum, $j = n$ in the second sum, and $j = n$ in the third sum.

$$\sum_{j+2=2}^{\infty} (j+2)(j+1)a_{j+2} x^j - \sum_{j=0}^{\infty} 2j a_j x^j + \sum_{j=0}^{\infty} 2a_j x^j = 0$$

$$\sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2} x^j - \sum_{j=0}^{\infty} 2j a_j x^j + \sum_{j=0}^{\infty} 2a_j x^j = 0$$

$$\sum_{j=0}^{\infty} \left[(j+2)(j+1)a_{j+2} - 2j a_j + 2a_j\right] x^j = 0$$

The quantity in square brackets must be zero.

$$(j+2)(j+1)a_{j+2} - 2j a_j + 2a_j = 0$$

Solve this equation for a_{j+2}.

$$a_{j+2} = \frac{2(j-1)}{(j+2)(j+1)} a_j$$

The first two coefficients, a_0 and a_1, are arbitrary. In order to make the second Hermite polynomial an odd function, a_0 is chosen to be zero and a_1 is chosen to be nonzero. Consequently, $a_2 = a_4 = \cdots = 0$; also, since $a_3 = 0$, $a_5 = a_7 = \cdots = 0$.

$$w(x) = a_1 x$$

By convention, the coefficient of the highest power of x is set to 2^1, so $a_1 = 2$. Therefore,

$$H_1(x) = 2x.$$

The Third Hermite Polynomial

To get the third Hermite polynomial $H_2(x)$, set $\lambda = 5$ and solve the resulting ODE.

$$w'' - 2xw' + 4w = 0$$

We seek a power series solution for w.

$$w(x) = \sum_{n=0}^{\infty} a_n x^n \quad \to \quad w'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \to \quad w''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these formulas into the ODE.

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=0}^{\infty} n a_n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

The first sum can be started from $n = 2$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

Substitute $j = n - 2$ in the first sum, $j = n$ in the second sum, and $j = n$ in the third sum.

$$\sum_{j+2=2}^{\infty} (j+2)(j+1) a_{j+2} x^j - \sum_{j=0}^{\infty} 2j a_j x^j + \sum_{j=0}^{\infty} 4 a_j x^j = 0$$

$$\sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j - \sum_{j=0}^{\infty} 2j a_j x^j + \sum_{j=0}^{\infty} 4 a_j x^j = 0$$

$$\sum_{j=0}^{\infty} \left[(j+2)(j+1) a_{j+2} - 2j a_j + 4a_j\right] x^j = 0$$

The quantity in square brackets must be zero.

$$(j+2)(j+1) a_{j+2} - 2j a_j + 4a_j = 0$$

Solve this equation for a_{j+2}.

$$a_{j+2} = \frac{2(j-2)}{(j+2)(j+1)} a_j$$

The first two coefficients, a_0 and a_1, are arbitrary. In order to make the third Hermite polynomial an even function, a_0 is chosen to be nonzero and a_1 is chosen to be zero. Consequently, $a_3 = a_5 = \cdots = 0$; also, since $a_4 = 0$, $a_6 = a_8 = \cdots = 0$.

$$w(x) = a_0 + a_2 x^2$$

By convention, the coefficient of the highest power of x is set to 2^2, so $a_2 = 4$. Setting $j = 0$ in the previous formula yields $a_2 = (-2)a_0$, which means $a_0 = -2$. Therefore,

$$H_2(x) = 4x^2 - 2.$$

The Fourth Hermite Polynomial

To get the fourth Hermite polynomial $H_3(x)$, set $\lambda = 7$ and solve the resulting ODE.

$$w'' - 2xw' + 6w = 0$$

We seek a power series solution for w.

$$w(x) = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad w'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad w''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these formulas into the ODE.

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=0}^{\infty} n a_n x^{n-1} + 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

The first sum can be started from $n = 2$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

Substitute $j = n - 2$ in the first sum, $j = n$ in the second sum, and $j = n$ in the third sum.

$$\sum_{j+2=2}^{\infty} (j+2)(j+1) a_{j+2} x^j - \sum_{j=0}^{\infty} 2j a_j x^j + \sum_{j=0}^{\infty} 6 a_j x^j = 0$$

$$\sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j - \sum_{j=0}^{\infty} 2j a_j x^j + \sum_{j=0}^{\infty} 6 a_j x^j = 0$$

$$\sum_{j=0}^{\infty} \left[(j+2)(j+1) a_{j+2} - 2j a_j + 6 a_j\right] x^j = 0$$

The quantity in square brackets must be zero.

$$(j+2)(j+1) a_{j+2} - 2j a_j + 6 a_j = 0$$

Solve this equation for a_{j+2}.

$$a_{j+2} = \frac{2(j-3)}{(j+2)(j+1)} a_j$$

The first two coefficients, a_0 and a_1, are arbitrary. In order to make the fourth Hermite polynomial an odd function, a_0 is chosen to be zero and a_1 is chosen to be nonzero. Consequently, $a_2 = a_4 = \cdots = 0$; also, since $a_5 = 0$, $a_7 = a_9 = \cdots = 0$.

$$w(x) = a_1 x + a_3 x^3$$

By convention, the coefficient of the highest power of x is set to 2^3, so $a_3 = 8$. Setting $j = 1$ in the previous formula yields $a_3 = -\frac{2}{3} a_1$, which means $a_1 = -12$. Therefore,

$$H_3(x) = 8x^3 - 12x.$$