Exercise 6

Show directly from the ODE (15) that the functions $H_k(x)e^{-x^2/2}$ are mutually orthogonal on the interval $(-\infty, \infty)$. That is

$$\int_{-\infty}^{\infty} H_k(x)H_l(x)e^{-x^2}\,dx = 0 \qquad \text{for } k \neq l.$$

(*Hint:* See Section 5.3.)

Solution

Equation (15) in the text is

$$v'' + (\lambda - x^2)v = 0. \tag{15}$$

If λ is a positive odd integer, then the solution for v is expressed in terms of Hermite polynomials. Suppose that $\lambda_k = 2k+1$ and $\lambda_l = 2l+1$ with $k = 0, 1, \ldots$ and $l = 0, 1, \ldots$ and $k \neq l$. The two ODEs for these values of λ are

$$v_k'' + (\lambda_k - x^2)v_k = 0 \tag{1}$$
$$v_l'' + (\lambda_l - x^2)v_l = 0, \tag{2}$$

and their solutions are

$$v_k(x) = H_k(x)e^{-x^2/2}$$
$$v_l(x) = H_l(x)e^{-x^2/2}.$$

Multiply both sides of equation (1) by v_l, and multiply both sides of equation (2) by v_k.

$$v_k''v_l + (\lambda_k - x^2)v_kv_l = 0$$
$$v_kv_l'' + (\lambda_l - x^2)v_kv_l = 0$$

Subtract both sides of the second equation from those of the first.

$$v_k''v_l - v_kv_l'' + (\lambda_k - \lambda_l)v_kv_l = 0$$

Solve the equation for v_kv_l.

$$v_k(x)v_l(x) = \frac{1}{\lambda_k - \lambda_l}(v_kv_l'' - v_k''v_l)$$

Integrate both sides with respect to x from $-\infty$ to ∞ and simplify the right side.

$$\begin{aligned}
\int_{-\infty}^{\infty} v_k(x)v_l(x)\,dx &= \int_{-\infty}^{\infty} \frac{1}{\lambda_k - \lambda_l}(v_kv_l'' - v_k''v_l)\,dx \\
&= \frac{1}{\lambda_k - \lambda_l}\left[\int_{-\infty}^{\infty} v_k(x)v_l''(x)\,dx - \int_{-\infty}^{\infty} v_k''(x)v_l(x)\,dx\right] \\
&= \frac{1}{\lambda_k - \lambda_l}\left[v_k(x)v_l'(x)\Big|_{-\infty}^{\infty} - \cancel{\int_{-\infty}^{\infty} v_k'(x)v_l'(x)\,dx} - v_k'(x)v_l(x)\Big|_{-\infty}^{\infty}\right. \\
&\qquad \left. + \cancel{\int_{-\infty}^{\infty} v_k'(x)v_l'(x)\,dx}\right]
\end{aligned}$$

Because of the decaying exponential functions in v_k and v_l, both terms tend to zero as $x \to \pm\infty$.

$$\begin{aligned}\int_{-\infty}^{\infty} v_k(x) v_l(x)\, dx &= \frac{1}{\lambda_k - \lambda_l}\left[\underbrace{v_k(x)}_{=\,0} v_l'(x)\bigg|_{-\infty}^{\infty} - v_k'(x)\underbrace{v_l(x)}_{=\,0}\bigg|_{-\infty}^{\infty}\right] \\ &= 0\end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} H_k(x) H_l(x) e^{-x^2}\, dx = 0 \qquad \text{for } k \neq l.$$