Exercise 1

Verify the formulas for the first three solutions of the hydrogen atom.

Solution

The first three solutions to the governing (Schrödinger) equation of the hydrogen atom,

$$iu_t = -\frac{1}{2}\Delta u - \frac{1}{r}u, \tag{1}$$

are

$$\begin{aligned}
u_1(r,t) &= e^{it/2}e^{-r} \\
u_2(r,t) &= e^{it/8}e^{-r/2}\left(1 - \frac{1}{2}r\right) \\
u_3(r,t) &= e^{it/18}e^{-r/3}\left(1 - \frac{2}{3}r + \frac{2}{27}r^2\right),
\end{aligned}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ represents the spherical coordinate. As these solutions are spherically symmetric, $u = u(r,t)$, equation (1) reduces to

$$\begin{aligned}
iu_t &= -\frac{1}{2}\left(u_{rr} + \frac{2}{r}u_r\right) - \frac{1}{r}u, \\
iu_t &= -\frac{1}{2}u_{rr} - \frac{1}{r}(u_r + u). \qquad (2)
\end{aligned}$$

<u>The First Solution</u>

The derivatives of $u_1(r,t)$ are as follows.

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= \frac{i}{2}e^{it/2}e^{-r} \\
\frac{\partial u_1}{\partial r} &= -e^{it/2}e^{-r} \\
\frac{\partial^2 u_1}{\partial r^2} &= e^{it/2}e^{-r}
\end{aligned}$$

Substitute these formulas into equation (2) to see if $u_1(r,t)$ is indeed a solution.

$$\begin{aligned}
i\left(\frac{i}{2}e^{it/2}e^{-r}\right) &\stackrel{?}{=} -\frac{1}{2}e^{it/2}e^{-r} - \frac{1}{r}\left(-e^{it/2}e^{-r} + e^{it/2}e^{-r}\right) \\
-\frac{1}{2}e^{it/2}e^{-r} &= -\frac{1}{2}e^{it/2}e^{-r}
\end{aligned}$$

Therefore, $u_1(r,t)$ satisfies the governing equation of the hydrogen atom.

The Second Solution

The derivatives of $u_2(r,t)$ are as follows.

$$\begin{aligned}
\frac{\partial u_2}{\partial t} &= \frac{i}{8}e^{it/8}e^{-r/2}\left(1-\frac{1}{2}r\right)\\
\frac{\partial u_2}{\partial r} &= -\frac{1}{2}e^{it/8}e^{-r/2}\left(2-\frac{1}{2}r\right)\\
\frac{\partial^2 u_2}{\partial r^2} &= \frac{1}{4}e^{it/8}e^{-r/2}\left(3-\frac{1}{2}r\right)
\end{aligned}$$

Substitute these formulas into equation (2) to see if $u_2(r,t)$ is indeed a solution.

$$\begin{aligned}
i\left[\frac{i}{8}e^{it/8}e^{-r/2}\left(1-\frac{1}{2}r\right)\right] &\stackrel{?}{=} -\frac{1}{2}\left[\frac{1}{4}e^{it/8}e^{-r/2}\left(3-\frac{1}{2}r\right)\right]-\frac{1}{r}\left[-\frac{1}{2}e^{it/8}e^{-r/2}\left(2-\frac{1}{2}r\right)+e^{it/8}e^{-r/2}\left(1-\frac{1}{2}r\right)\right]\\
-\frac{1}{8}e^{it/8}e^{-r/2}\left(1-\frac{1}{2}r\right) &\stackrel{?}{=} -\frac{1}{8}e^{it/8}e^{-r/2}\left(3-\frac{1}{2}r\right)-\frac{1}{r}\left[e^{it/8}e^{-r/2}\left(-\frac{r}{4}\right)\right]\\
-\frac{1}{8}e^{it/8}e^{-r/2}\left(1-\frac{1}{2}r\right) &\stackrel{?}{=} -\frac{1}{8}e^{it/8}e^{-r/2}\left(3-\frac{1}{2}r\right)-\frac{1}{8}e^{it/8}e^{-r/2}(-2)\\
-\frac{1}{8}e^{it/8}e^{-r/2}\left(1-\frac{1}{2}r\right) &= -\frac{1}{8}e^{it/8}e^{-r/2}\left(1-\frac{1}{2}r\right)
\end{aligned}$$

Therefore, $u_2(r,t)$ satisfies the governing equation of the hydrogen atom.

The Third Solution

The derivatives of $u_3(r,t)$ are as follows.

$$\begin{aligned}
\frac{\partial u_3}{\partial t} &= \frac{i}{18}e^{it/18}e^{-r/3}\left(1-\frac{2}{3}r+\frac{2}{27}r^2\right)\\
\frac{\partial u_3}{\partial r} &= -\frac{1}{3}e^{it/18}e^{-r/3}\left(3-\frac{10}{9}r+\frac{2}{27}r^2\right)\\
\frac{\partial^2 u_3}{\partial r^2} &= \frac{1}{9}e^{it/18}e^{-r/3}\left(\frac{19}{3}-\frac{14}{9}r+\frac{2}{27}r^2\right)
\end{aligned}$$

Substitute these formulas into equation (2) to see if $u_3(r,t)$ is indeed a solution.

$$\begin{aligned}
i\left[\frac{i}{18}e^{it/18}e^{-r/3}\left(1-\frac{2}{3}r+\frac{2}{27}r^2\right)\right] &\stackrel{?}{=} -\frac{1}{2}\left[\frac{1}{9}e^{it/18}e^{-r/3}\left(\frac{19}{3}-\frac{14}{9}r+\frac{2}{27}r^2\right)\right]\\
&\qquad -\frac{1}{r}\left[-\frac{1}{3}e^{it/18}e^{-r/3}\left(3-\frac{10}{9}r+\frac{2}{27}r^2\right)\right.\\
&\qquad\qquad \left.+e^{it/18}e^{-r/3}\left(1-\frac{2}{3}r+\frac{2}{27}r^2\right)\right]\\
-\frac{1}{18}e^{it/18}e^{-r/3}\left(1-\frac{2}{3}r+\frac{2}{27}r^2\right) &\stackrel{?}{=} -\frac{1}{18}e^{it/18}e^{-r/3}\left(\frac{19}{3}-\frac{14}{9}r+\frac{2}{27}r^2\right)\\
&\qquad -\frac{1}{r}\left[e^{it/18}e^{-r/3}\left(-\frac{8}{27}r+\frac{4}{81}r^2\right)\right]
\end{aligned}$$

$$-\frac{1}{18}e^{it/18}e^{-r/3}\left(1-\frac{2}{3}r+\frac{2}{27}r^2\right) \stackrel{?}{=} -\frac{1}{18}e^{it/18}e^{-r/3}\left(\frac{19}{3}-\frac{14}{9}r+\frac{2}{27}r^2\right) - \frac{1}{18}e^{it/18}e^{-r/3}\left(-\frac{16}{3}+\frac{8}{9}r\right)$$

$$-\frac{1}{18}e^{it/18}e^{-r/3}\left(1-\frac{2}{3}r+\frac{2}{27}r^2\right) = -\frac{1}{18}e^{it/18}e^{-r/3}\left(1-\frac{2}{3}r+\frac{2}{27}r^2\right)$$

Therefore, $u_3(r,t)$ satisfies the governing equation of the hydrogen atom.