Exercise 2.1.5

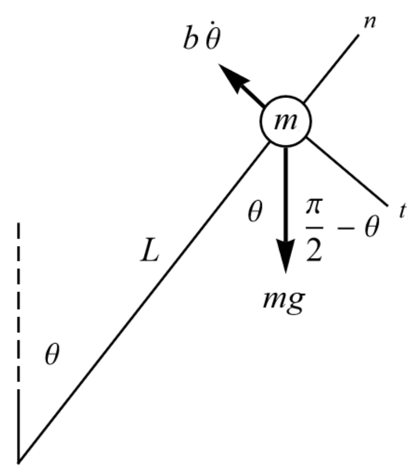
(A mechanical analog)

(a) Find a mechanical system that is approximately governed by $\dot{x} = \sin x$.

(b) Using your physical intuition, explain why it now becomes obvious that $x^* = 0$ is an unstable fixed point and $x^* = \pi$ is stable.

Solution

Draw a free-body diagram for an inverted pendulum in a viscous medium such that the resistive force is proportional to velocity.



According to Newton's second law,

$$\mathbf{F} = m\mathbf{a}.$$

Consider the sum of the forces in the tangential direction t.

$$\sum F_t = ma_t \quad \rightarrow \quad mg\cos\left(\frac{\pi}{2} - \theta\right) - b\dot{\theta} = m(L\alpha_t) \quad \rightarrow \quad mg\sin\theta - b\dot{\theta} = mL\ddot{\theta}$$

Assume that the medium is so viscous that the acceleration of the pendulum is negligible.

$$mg\sin\theta - b\dot{\theta} \approx 0$$

Solve for $\dot{\theta}$.

$$\dot{\theta} = \frac{mg}{b}\sin\theta$$

With a certain choice of units, $mg/b = 1$.

$$\dot{\theta} = \sin\theta$$

If $\theta = 0$ or $\theta = \pi$, then there's no tangential force, so a mass that starts at $\theta = 0$ or $\theta = \pi$ initially remains at rest by Newton's first law. $\theta = 0$ is an unstable fixed point because the smallest nudge can send the mass into motion, which will never return to equilibrium at $\theta = 0$. On the other hand, $\theta = \pi$ is a stable fixed point because the same nudge will not change the mass's equilibrium position (due to damping by the viscous medium).