Exercise 2.2.3

Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions. Then try for a few minutes to obtain the analytical solution for $x(t)$; if you get stuck, don't try for too long since in several cases it's impossible to solve the equation in closed form!

$$\dot{x} = x - x^3$$

Solution

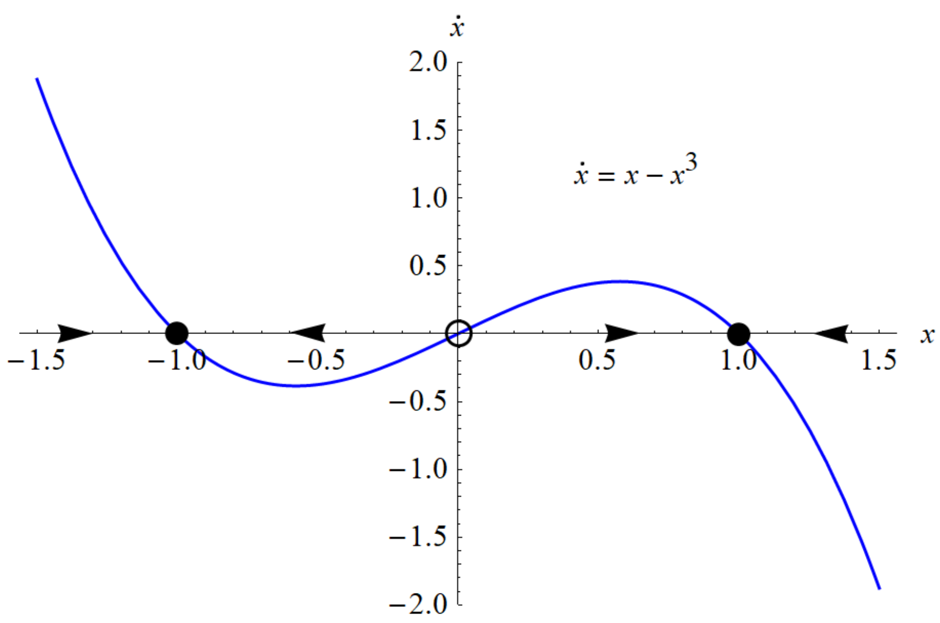
The fixed points of the flow occur where $\dot{x} = 0$.

$$x^* - x^{*3} = 0$$

$$x^*(1 - x^{*2}) = 0$$

$$x^* = \{-1, 0, 1\}$$

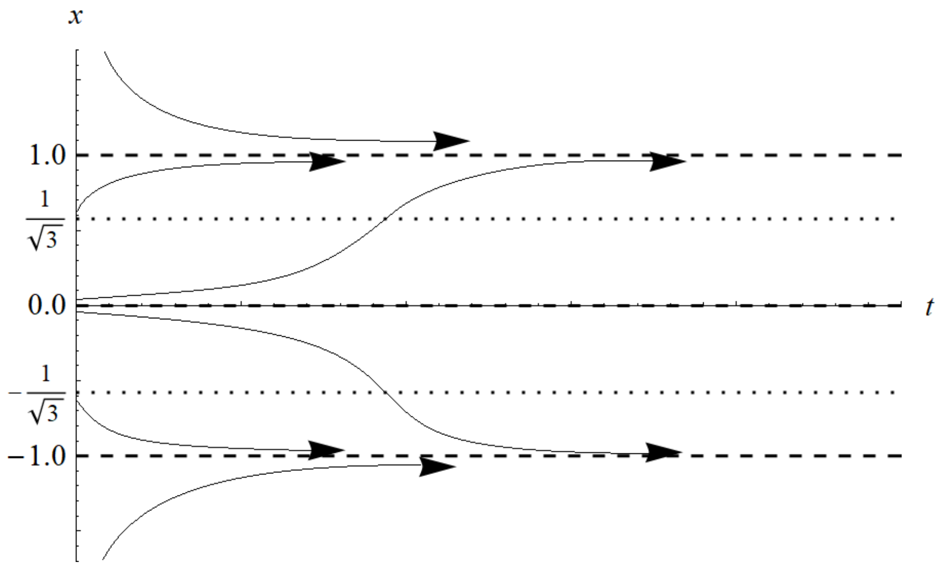
Plot the graph of $\dot{x}$ versus x to determine whether each is stable or unstable.



When the function is negative the flow moves to the left, and when the function is positive the flow moves to the right. Therefore, $x^* = -1$ is locally unstable, and $x^* = 1$ is locally stable. Observe that the maxima of this curve occur at

$$1 - 3x^2 = 0 \quad \rightarrow \quad x = \pm\frac{1}{\sqrt{3}}.$$

A qualitative sketch of x versus t is shown for various initial conditions.



The aim now is to solve the following initial value problem.

$$\frac{dx}{dt} = x - x^3, \quad x(0) = x_0$$

Solve the ODE by separating variables and using partial fraction decomposition.

$$\frac{dx}{x - x^3} = dt$$

$$\int \frac{dx}{x(1+x)(1-x)} = \int dt$$

$$\int \left(\frac{1}{x} - \frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{2}}{x-1} \right) dx = t + C$$

$$\ln|x| - \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| = t + C$$

$$2\ln|x| - \ln|(x+1)(x-1)| = 2t + 2C$$

$$\ln\left|\frac{x^2}{(x+1)(x-1)}\right| = 2t + D \tag{1}$$

Now apply the initial condition to determine D.

$$\ln\left|\frac{x_0^2}{(x_0+1)(x_0-1)}\right| = D$$

As a result, equation (1) becomes

$$\ln\left|\frac{x^2}{(x+1)(x-1)}\right| = 2t + \ln\left|\frac{x_0^2}{(x_0+1)(x_0-1)}\right|.$$

Bring the logarithms to the left side and combine them.

$$\ln\left|\frac{x^2}{(x+1)(x-1)}\cdot\frac{(x_0+1)(x_0-1)}{x_0^2}\right| = 2t$$

$$\left|\frac{x^2}{(x+1)(x-1)}\cdot\frac{(x_0+1)(x_0-1)}{x_0^2}\right| = e^{2t}$$

$$\frac{x^2}{(x+1)(x-1)}\cdot\frac{(x_0+1)(x_0-1)}{x_0^2} = \pm e^{2t}$$

Choose the plus sign so that the equation is a true statement when $x = x_0$ and $t = 0$.

$$\frac{x^2}{(x+1)(x-1)}\cdot\frac{(x_0+1)(x_0-1)}{x_0^2} = e^{2t}$$

Solve for x.

$$\frac{x^2}{x^2-1} = \frac{x_0^2}{x_0^2-1}e^{2t}$$

$$\frac{x^2-1}{x^2} = \frac{x_0^2-1}{x_0^2}e^{-2t}$$

$$1 - \frac{1}{x^2} = \frac{(x_0^2-1)e^{-2t}}{x_0^2}$$

$$\frac{1}{x^2} = 1 - \frac{(x_0^2-1)e^{-2t}}{x_0^2}$$

$$\frac{1}{x^2} = \frac{x_0^2-(x_0^2-1)e^{-2t}}{x_0^2}$$

$$x(t) = \pm\frac{x_0}{\sqrt{x_0^2-(x_0^2-1)e^{-2t}}}$$

Again, choose the plus sign so that the equation is a true statement when $x = x_0$ and $t = 0$. Therefore,

$$x(t) = \frac{x_0}{\sqrt{x_0^2-(x_0^2-1)e^{-2t}}}.$$