

**Exercise 2.3.1**

(Exact solution of logistic equation) There are two ways to solve the logistic equation  $\dot{N} = rN(1 - N/K)$  analytically for an arbitrary initial condition  $N_0$ .

- Separate variables and integrate, using partial fractions.
- Make the change of variables  $x = 1/N$ . Then derive and solve the resulting differential equation for  $x$ .

**Solution****Part (a)**

Solve the logistic equation by separating variables and integrating both sides.

$$\begin{aligned}\frac{dN}{dt} &= \frac{r}{K}N(K - N) \\ \frac{dN}{N(K - N)} &= \frac{r}{K} dt \\ \int \frac{dN}{N(K - N)} &= \int \frac{r}{K} dt \\ \int \left( \frac{1}{N} - \frac{1}{N - K} \right) dN &= \int \frac{r}{K} dt \\ \frac{1}{K} \ln |N| - \frac{1}{K} \ln |N - K| &= \frac{r}{K}t + C \\ \ln |N| - \ln |N - K| &= rt + CK \\ \ln \left| \frac{N}{N - K} \right| &= rt + CK \\ \left| \frac{N}{N - K} \right| &= e^{rt+CK}\end{aligned}$$

Place  $\pm$  on the right side in order to remove the absolute value sign.

$$\frac{N}{N - K} = \pm e^{CK} e^{rt}$$

Use a new constant  $A$  for  $\pm e^{CK}$ .

$$\frac{N}{N - K} = Ae^{rt} \tag{1}$$

Now apply the initial condition  $N(0) = N_0$  to determine  $A$ .

$$\frac{N_0}{N_0 - K} = A$$

As a result, equation (1) becomes

$$\begin{aligned}\frac{N}{N-K} &= \frac{N_0}{N_0-K} e^{rt} \\ N &= N \frac{N_0}{N_0-K} e^{rt} - K \frac{N_0}{N_0-K} e^{rt} \\ \left(1 - \frac{N_0}{N_0-K} e^{rt}\right) N &= -K \frac{N_0}{N_0-K} e^{rt} \\ \frac{N_0 - K - N_0 e^{rt}}{N_0 - K} N &= -\frac{K N_0 e^{rt}}{N_0 - K}.\end{aligned}$$

Multiply both sides by  $N_0 - K$ .

$$[N_0(1 - e^{rt}) - K]N = -K N_0 e^{rt}$$

Therefore,

$$N(t) = \frac{K N_0 e^{rt}}{K - N_0(1 - e^{rt})}.$$

**Part (b)**

The same initial value problem will be solved here but with a different method.

$$\frac{dN}{dt} = rN \left( 1 - \frac{N}{K} \right), \quad N(0) = N_0$$

Make the change of variables,

$$x = \frac{1}{N},$$

in the ODE and the initial condition.

$$\frac{d}{dt} \left( \frac{1}{x} \right) = \frac{r}{x} \left( 1 - \frac{1}{Kx} \right), \quad x(0) = \frac{1}{N(0)} = \frac{1}{N_0}$$

Simplify the transformed ODE.

$$\begin{aligned} \left( -\frac{1}{x^2} \right) \frac{dx}{dt} &= \frac{r}{x} \left( 1 - \frac{1}{Kx} \right) \\ \frac{dx}{dt} &= r \left( \frac{1}{K} - x \right) \\ \frac{dx}{dt} + rx &= \frac{r}{K} \end{aligned} \tag{2}$$

This is a first-order linear ODE, so it can be solved with an integrating factor  $I$ .

$$I = \exp \left( \int^t r \, ds \right) = e^{rt}$$

Multiply both sides of equation (2) by  $I$ .

$$e^{rt} \frac{dx}{dt} + r e^{rt} x = \frac{r}{K} e^{rt}$$

Rewrite the left side as a derivative using the product rule.

$$\frac{d}{dt} (e^{rt} x) = \frac{r}{K} e^{rt}$$

Integrate both sides with respect to  $t$ .

$$e^{rt} x = \frac{1}{K} e^{rt} + D \tag{3}$$

Apply the initial condition now to determine  $D$ .

$$\frac{1}{N_0} = \frac{1}{K} + D \quad \rightarrow \quad D = \frac{1}{N_0} - \frac{1}{K} = \frac{K - N_0}{KN_0}$$

Consequently, equation (3) becomes

$$e^{rt} x = \frac{1}{K} e^{rt} + \frac{K - N_0}{KN_0}.$$

Solve for  $x$ .

$$\begin{aligned}x(t) &= \frac{1}{K} + \frac{K - N_0}{KN_0e^{rt}} \\&= \frac{N_0e^{rt}}{KN_0e^{rt}} + \frac{K - N_0}{KN_0e^{rt}} \\&= \frac{N_0e^{rt} + K - N_0}{KN_0e^{rt}} \\&= \frac{K + N_0(e^{rt} - 1)}{KN_0e^{rt}} \\&= \frac{K - N_0(1 - e^{rt})}{KN_0e^{rt}}\end{aligned}$$

Therefore, since  $N(t) = 1/x(t)$ ,

$$N(t) = \frac{KN_0e^{rt}}{K - N_0(1 - e^{rt})}.$$